MA 30300
FINAL EXAM INSTRUCTIONS

NAME ______________________ INSTRUCTOR ______________________

1. Instructor's names: Chen, Dong, Howard, or Lundberg

2. Course number: MA30300.

3. SECTION NUMBERS:

0061 for MWF 10:30AM-11:20AM REC 313 class by Erik Lundberg
0071 for MWF 10:30AM-11:20AM MATH 175 class by Min Chen
0072 for MWF 11:30AM-12:20AM REC 313 class by Erik Lundberg
0073 for TR 01:30PM-02:45PM UNIV 117 class by Suchuan Dong
0074 for TR 12:00AM-01:15PM UNIV 117 class by Suchuan Dong
0075 for MWF 01:30PM-02:20PM UNIV 219 class by Ha Howard
0076 for MWF 12:30AM-01:20PM UNIV 219 class by Ha Howard

4. TEST/QUIZ NUMBER is:

01 if this sheet is **yellow**
02 if this sheet is **blue**
03 if this sheet is **white**

5. Sign the scantron sheet.

6. Laplace transform table and a formula sheet on Fourier series and some PDEs are provided at the end of this booklet.

7. There are 20 questions, each worth 5 points. Do all your work on the question sheets. Turn in both the scantron form and the question sheets when you are done.

8. Show your work on the question sheets. Although no partial credit will be given, any disputes about grades or grading will be settled by examining your written work on the question sheets.

9. NO CALCULATORS, BOOKS, OR PAPERS ARE ALLOWED. Use the back of the test pages for scrap paper.

1. Consider the Euler equation

$$x^2y'' + \alpha xy' + \beta y = 0$$

Find all among the following conditions on α and β so that all solutions are bounded as $x \to 0$?

A. $\alpha < 1,\ \beta \geq 0.$

B. $\alpha = 1,\ \beta > 0.$

C. $\alpha = 1,\ \beta \geq 0$

D. A and C

E. $\boxed{\text{A and B}}$

2. Find the value of

$$\sum_{n=1, n=odd}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots .$$

Hint: Evaluate the following Fourier series at $x = 0$ using the fact that the Fourier series of period 4 for the function $f(x) = 2 - |x|$ defined on $(-2, 2)$ is

$$1 + \frac{8}{\pi^2} \sum_{n=1, n=odd}^{\infty} \frac{\cos(\frac{n\pi x}{2})}{n^2}.$$

A. $\dfrac{\pi}{3}$

B. $1 + \dfrac{8}{\pi^2}$

C. $\boxed{\dfrac{\pi^2}{8}}$

D. $\dfrac{\pi^2 + 1}{9}$

E. $\dfrac{8}{\pi^2}$

3. Which of the following (if any) is not an eigenvalue-eigenfunction pair for the eigenvalue problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(3\pi) = 0?$$

A. $\{\lambda = \frac{4}{9}, y(x) = \cos(\frac{2x}{3})\}$

B. $\boxed{\{\lambda = \frac{1}{4}, y(x) = \cos(\frac{x}{2})\}}$

C. $\{\lambda = 4, y(x) = \cos(2x)\}$

D. $\{\lambda = 9, y(x) = \cos(3x)\}$

E. These are all valid eigenvalue-eigenfunction pairs.

4. Which is the solution for

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 \le x \le 1,\ y \ge 0$$
$$u(0,y) = 0, \qquad u(1,y) = 0$$
$$u(x,0) = 2\sin(3\pi x), \qquad \lim_{y\to\infty} u(x,y) = 0?$$

A. $2e^{-3\pi y}\sinh(3\pi x)$

B. $\boxed{2e^{-3\pi y}\sin(3\pi x)}$

C. $2\sinh(3\pi y)\sin(3\pi x)$

D. $2\sinh(3\pi y)\cos(3\pi x)$

E. none of the above.

5. If $y'' + 3y' + 2y = \delta(t-2)$ with $y(0) = 1$, $y'(0) = -1$, then $y(4)$ is

A. $e^{-2} - e^{-4}$

B. $e^{-2} - 2e^{-4}$

C. $e^{-4} - e^{-6}$

D. $\boxed{e^{-2}}$

E. $2e^{-4} - e^{-2}$

6. If $y(t)$ solves the initial value problem

$$y'' + y = \begin{cases} t/2, & 0 \le t < 6 \\ 3, & 6 \le t \end{cases}, \qquad y(0) = 0, \quad y'(0) = 0$$

for $t \ge 0$, then

A. $y = \begin{cases} \frac{1}{2}\sin(t) + \frac{1}{2}t & t < 6 \\ \frac{1}{2}\sin(t) + 3 & t \ge 6 \end{cases}$

B. $y = \begin{cases} \frac{1}{2}\sin(t) + \frac{1}{2}t & t < 6 \\ \sin(t) + 3 & t \ge 6 \end{cases}$

C. $y = \begin{cases} -\frac{1}{2}\sin(t) + \frac{1}{2}t & t < 6 \\ \frac{1}{2}\sin(t) + \frac{1}{2}\sin(t-6) + 3 & t \ge 6 \end{cases}$

D. $y = \begin{cases} \frac{1}{2}\cos(t) + \frac{1}{2}t & t < 6 \\ \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t-6) + 3 & t \ge 6 \end{cases}$

E. $\boxed{y = \begin{cases} -\frac{1}{2}\sin(t) + \frac{1}{2}t & t < 6 \\ -\frac{1}{2}\sin(t) + \frac{1}{2}\sin(t-6) + 3 & t \ge 6 \end{cases}}$

7. If

$$\mathcal{L}\{f(t)\} = \frac{8s^2 - 4s + 12}{s(s^2+4)},$$

then what is the value of $f(\frac{\pi}{2})$?

A. 6

B. 3

C. 1

D. $\boxed{\text{-2}}$

E. -4

8. The indicial equation for $2x^2y'' - xy' + (1+x)y = 0$ at $x_0 = 0$ is

A. $\boxed{2r^2 - 3r + 1 = 0}$

B. $2r^2 - 3r = 0$

C. $r^2 - 3r + 1 = 0$

D. $r^2 - r + 1 = 0$

E. $2r^2 - r + 1 = 0$

9. Consider the system

$$\mathbf{x}' = A\mathbf{x},$$

and suppose the eigenvalues are $r = -2 \pm 3i$. Then classify the origin in terms of the phase portrait.

A. $\boxed{\text{spiral}}$

B. center

C. proper node

D. saddle

E. improper node

10. Find the Laplace transform of

$$f(t) = \begin{cases} 0 & \text{when } t < \pi, \\ t - \pi & \text{when } \pi \le t < 2\pi, \\ 0 & \text{when } t \ge 2\pi. \end{cases}$$

A. $\boxed{e^{-\pi s}\frac{1}{s^2} - e^{-2\pi s}\frac{1}{s^2} - \pi e^{-2\pi s}\frac{1}{s}}$

B. $e^{-\pi s}\frac{1}{s^2} - e^{-2\pi s}\frac{1}{s^2}$

C. $e^{\pi s}\frac{1}{s^2} - e^{2\pi s}\frac{1}{s^2} - \pi e^{2\pi s}\frac{1}{s^2}$

D. $\frac{1}{s}\left(e^{-\pi s} - e^{-2\pi s}\right)$

E. $\frac{1}{2}e^{-\pi s}\frac{1}{s^2} - \frac{1}{3}e^{-2\pi s}\frac{1}{s^2} - \pi e^{-2\pi s}\frac{1}{s}$

11. Suppose $X(t)$ is a fundamental matrix for the homogeneous equation $\mathbf{x}' = A\mathbf{x}$, and suppose that $\mathbf{x} = X(t)\mathbf{c}(t)$ is a solution of the nonhomogeneous equation $\mathbf{x}' = A\mathbf{x} + g(t)$. Which of the following is a correct equation for $\mathbf{c}$?

A. $\mathbf{c}' = Xg$

B. $\boxed{\mathbf{c}' = X^{-1}g}$

C. $\mathbf{c}' = Xg'$

D. $\mathbf{c}' = X'g + Xg'$

E. $\mathbf{c} = X^{-1}g'$

12. Consider the system:

$$\mathbf{x}' = \begin{bmatrix} 1 & c \\ 1 & -2 \end{bmatrix} \mathbf{x}.$$

For what values of c is the origin an asymptotically stable spiral?

A. $c < 0$

B. $-5/2 < c < 2$

C. $c < -1$

D. $-10 < c < 3/2$

E. $\boxed{c < -9/4}$

13. (**Hint**: By observing the correct form of the solution, you should be able to find the answer for this problem)

The matrix $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ has eigenvalues 1 and -1, and their corresponding eigenvectors are respectively $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Which of the following is the solution $\mathbf{X}$ to the initial value problem

$$\mathbf{X}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\mathbf{X} + \begin{bmatrix} e^t \\ 5e^t \end{bmatrix}, \qquad \mathbf{X}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A. $\begin{bmatrix} -e^t + e^{-t} + te^t \\ 2e^t - 2e^{-t} + te^t \end{bmatrix}$

B. $\begin{bmatrix} 3e^t + e^{-t} - te^t \\ e^t - e^{-t} + te^t \end{bmatrix}$

C. $\begin{bmatrix} -e^t + e^{-t} - 2te^t \\ -2e^t + 2e^{-t} - 2te^t \end{bmatrix}$

D. $\boxed{\begin{bmatrix} e^t - e^{-t} - te^t \\ 3e^t - 3e^{-t} - te^t \end{bmatrix}}$

E. $\begin{bmatrix} e^t - e^{-t} - 2te^t \\ -3e^t + 3e^{-t} - 2te^t \end{bmatrix}$

14. The general solution to the system $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 4t \\ -2e^t \end{pmatrix}$ is

A. $\mathbf{x} = c_1e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -2t^2 \\ -2e^t \end{pmatrix}$

B. $\mathbf{x} = c_1e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -2t \\ 4e^t \end{pmatrix}$

C. $\boxed{\mathbf{x} = c_1e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -2t - 1 \\ e^t \end{pmatrix}}$

D. $\mathbf{x} = c_1e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -2t - 1 \\ te^t \end{pmatrix}$

E. $\mathbf{x} = c_1e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -2t + 1 \\ 4e^t \end{pmatrix}$

15. For which value of λ does the following equation has nontrivial solution

$$\begin{cases} u_{xx} + u_{yy} + \lambda u = 0 & 0 \leq x \leq a, 0 \leq y \leq b \\ u(0, y) = 0 = u(a, y) & 0 < y < b \\ u(x, 0) = 0 = u(x, b) & 0 < x < a \end{cases}$$

Find the corresponding solution.

Hint: Use separation of variables.

A. The possible values of λ are : $\dfrac{m^2\pi^2}{a^2}$, $m = 1, 2, \ldots$ with corresponding solution

$$\sin(\frac{m\pi x}{a}) \sin(\frac{m\pi y}{a}); \quad m = 1, 2, \ldots$$

B. The possible values of λ are : $\dfrac{n^2\pi^2}{b^2}$, $n = 1, 2, \ldots$ with corresponding solution

$$\sin(\frac{m\pi x}{b}) \sin(\frac{m\pi y}{b}) \quad n = 1, 2, \ldots$$

C. $\boxed{\text{The possible values of } \lambda \text{ are: } \dfrac{m^2\pi^2}{a^2} + \dfrac{n^2\pi^2}{b^2};\ n = 1, 2, \ldots, m = 1, 2, \ldots}$ with corresponding solution

$$u_{mn} = \sin(\frac{m\pi x}{a}) \sin(\frac{m\pi y}{a}); \quad n = 1, 2, \ldots, m = 1, 2, \ldots$$

D. There are no such value of λ.

16. For the equation $x^2y'' + \frac{7}{6}xy' + (x^2 - \frac{1}{6})y = 0$, the correct forms for 2 linearly independent solutions with $x > 0$ are

A. $\boxed{y_1 = x^{-\frac{1}{2}}(1 + \sum_{n=1}^{\infty} a_n x^n), y_2 = x^{\frac{1}{3}}(1 + \sum_{n=1}^{\infty} b_n x^n)}$

B. $y_1 = x^{-1}(1 + \sum_{n=1}^{\infty} a_n x^n)$,
$y_2 = y_1(x)\ln x + x^{-1}(1 + \sum_{n=1}^{\infty} b_n x^n)$

C. $y_1 = x^{-1/2}(1 + \sum_{n=1}^{\infty} a_n x^n)$,
$y_2 = x^{-1/2}(1 + \sum_{n=1}^{\infty} b_n x^n)$

D. $y_1 = x(1 + \sum_{n=1}^{\infty} a_n x^n)$,
$y_2 = y_1(x)\ln x + x(1 + \sum_{n=1}^{\infty} b_n x^n)$

E. $y_1 = \sqrt{x}\cos(\sqrt{3}\ln x/2)(1 + \sum_{n=1}^{\infty} a_n x^n)$,
$y_2 = \sqrt{x}\sin(\sqrt{3}\ln x/2)(1 + \sum_{n=1}^{\infty} b_n x^n)$

17. Given

$$f(t) = \begin{cases} 0 & t < 1 \\ t & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}$$

then what is the value of $\mathcal{L}\{f(t)\}$?

A. $\frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}$

B. $\frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$

C. $\boxed{\frac{e^{-s}}{s^2} + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2} - \frac{2e^{-2s}}{s}}$

D. $\frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2}$

E. $\frac{1}{s^2} + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$

18.

$$y'' + 2y' + 2y = \delta(t - \frac{\pi}{6}) \sin t$$
$$y(0) = 0$$
$$y'(0) = 1$$

Then $y\left(\frac{\pi}{2}\right) =$ ________.

A. $\boxed{e^{-\frac{\pi}{2}}}$
B. $\frac{3}{2}e^{-\frac{\pi}{2}}$
C. $\frac{1}{2}e^{-\frac{\pi}{3}}$
D. $e^{-\frac{\pi}{2}} + \frac{1}{4}e^{-\frac{\pi}{3}}$
E. $e^{-\frac{\pi}{2}} + \frac{\sqrt{3}}{4}e^{-\frac{\pi}{3}}$

19. Let $u(x,t)$ satisfy the heat equation

$$u_{xx} = u_t, \quad 0 < x < 1, \quad t > 0$$
$$u(0,t) = 0, \quad u(1,t) = 2$$
$$u(x,0) = 2x + \sin(\pi x).$$

Then $u(\frac{1}{2}, 1) =$

A. 1
B. 2
C. $\boxed{1 + e^{-\pi^2}}$
D. $1 - \dfrac{1}{2}e^{-\pi^2}$
E. $1 + 2e^{-\pi^2}$

20. The approximate value of the solution at $t = 2$ of

$$y' = 1 - t + 6y, \quad y(1) = 2$$

is evaluated using the Backward Euler method with $h = 1$.
Recall: The formula of Backward Euler method is

$$t_{n+1} = t_n + h; \quad y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}) \quad \text{for} \quad n = 0, 1, 2, \cdots.$$

The approximate value of $y(2)$ is

A. $-\frac{1}{2}$
B. $-\frac{1}{3}$
C. $\boxed{-\frac{1}{5}}$
D. $-\frac{1}{8}$
E. $\frac{1}{3}$

Formula sheet

<u>**Fourier series:**</u> For a $2L$-periodic function $f(x)$, the Fourier series for f is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L},$$

where for $n = 1, 2, \cdots$,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx.$$

<u>**Heat equation 1:**</u> The solution of the heat equation $\alpha^2 u_{xx} = u_t$, $0 < x < L$, $t > 0$, satisfying the (fixed temperature) homogeneous boundary conditions $u(0,t) = u(L,t) = 0$ for $t > 0$ with initial temperature $u(x,0) = f(x)$ has the general form

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin\frac{n\pi x}{L}, \quad \text{where} \quad c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx.$$

<u>**Heat equation 2:**</u> The solution of the heat equation $\alpha^2 u_{xx} = u_t$, $0 < x < L$, $t > 0$, satisfying the insulated boundary conditions $u_x(0,t) = u_x(L,t) = 0$ for $t > 0$ with initial temperature $u(x,0) = f(x)$ has the general form

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \cos\frac{n\pi x}{L}, \quad \text{where} \quad c_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx.$$

<u>**Wave equation:**</u> The solution of the wave equation $\alpha^2 u_{xx} = u_{tt}$, $0 < x < L$, $t > 0$, satisfying the homogeneous boundary conditions $u(0,t) = u(L,t) = 0$ for $t > 0$ and initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ for $0 \le x \le L$ has the general form

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(c_n \cos\frac{n\pi\alpha t}{L} + k_n \sin\frac{n\pi\alpha t}{L}\right)$$

where

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \quad \text{and} \quad k_n = \frac{2}{n\pi\alpha}\int_0^L g(x)\sin\frac{n\pi x}{L}dx.$$

<u>**Laplace equation:**</u> The solution of the Laplace equation $u_{xx} + u_{yy} = 0$, $0 < x < a$, $0 \le y \le b$, satisfying the boundary conditions $u(x,0) = u(x,b) = 0$ for $0 < x < a$ and $u(0,y) = 0$ and $u(a,y) = f(y)$ for $0 \le y \le b$ has the general form

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\frac{n\pi x}{b} \sin\frac{n\pi y}{b} \quad \text{where} \quad c_n = \frac{2}{b\sinh(\frac{n\pi a}{b})}\int_0^b f(y)\sin\frac{n\pi y}{b}dy.$$

Figure 1: Laplace Transform Table

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$
2.	e^{at}	$\frac{1}{s-a}$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$t^p \ (p > -1)$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sin at$	$\frac{a}{s^2+a^2}$
6.	$\cos at$	$\frac{s}{s^2+a^2}$
7.	$\sinh at$	$\frac{a}{s^2-a^2}$
8.	$\cosh at$	$\frac{s}{s^2-a^2}$
9.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
10.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
11.	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
12.	$u_c(t)$	$\frac{e^{-cs}}{s}$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right) \ \ c > 0$
16.	$\int_0^t f(t-\tau)\,g(\tau)\,d\tau$	$F(s)\,G(s)$
17.	$\delta(t-c)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$
19.	$(-t)^n f(t)$	$F^{(n)}(s)$