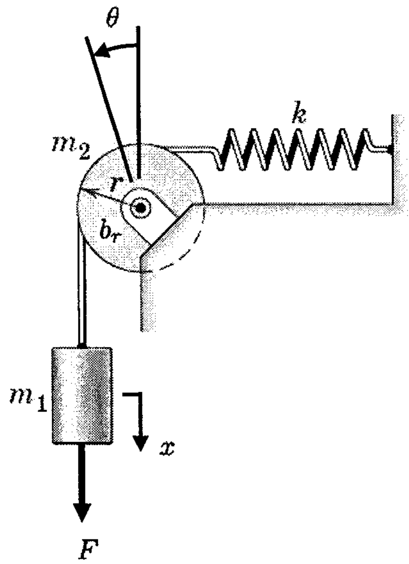
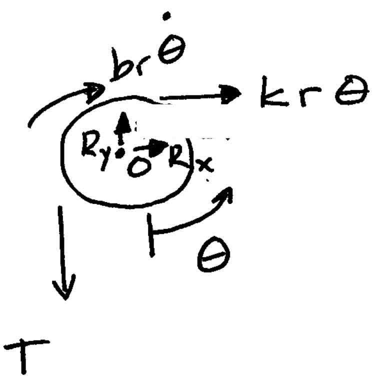
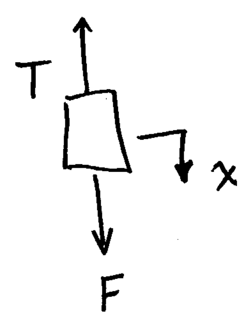
Problem 1. **(40%)** The system shown below consists of a mass and pulley, the latter of which has rotational inertia $I = m_2 r^2/2$. Assuming that there is no gravity, no slip between the rope and the pulley, and torsional viscous dissipation at the center of the pulley with coefficient b_r:
(a) Draw complete Free Body Diagrams (FBDs) for all basic elements.
(b) Derive the equation of motion governing the system's behavior in terms of the displacement x.
(c) Using x as an output and F as the input, derive the transfer function which governs the system's forced response.



(a)





(b)

$$+\circlearrowleft \sum M_o = I_o \ddot{\theta} \Rightarrow$$

$$Tr - kr^2\theta - b_r\dot{\theta} = \frac{1}{2} m_2 r^2 \ddot{\theta} \quad (1)$$

$$+\downarrow \sum F_x = m_1 \ddot{x} \Rightarrow$$

$$F - T = m_1 \ddot{x} \quad (2)$$

Kinematics: $x \overset{(1)}{=} r\theta \Rightarrow \ddot{x} = r\ddot{\theta} \quad (3)$

$\dot{x} = r\dot{\theta}$

This page is for extra work related to Problem 1.

$$(1) \Rightarrow T = kr\theta + b_r\frac{\dot{\theta}}{r} + \frac{1}{2}m_2 r\ddot{\theta}$$

Sub into (2) $\Rightarrow$

$$F - kr\theta - b_r\frac{\dot{\theta}}{r} - \frac{1}{2}m_2 r\ddot{\theta} = m_1\ddot{x}$$

Applying (3) $\Rightarrow$

$$F - kx - b_r\frac{\dot{x}}{r^2} - \frac{1}{2}m_2\ddot{x} = m_1\ddot{x}$$

$$\Rightarrow \left(m_1 + \frac{1}{2}m_2\right)\ddot{x} + \left(\frac{b_r}{r^2}\right)\dot{x} + kx = F \qquad (4)$$

(c) $\mathcal{L}\{(4)\} \Rightarrow$

$$\left(m_1 + \frac{1}{2}m_2\right)s^2X(s) + \left(\frac{b_r}{r^2}\right)sX(s) + kX(s) = F(s)$$

$$\Rightarrow G(s) = \frac{X(s)}{F(s)} = \frac{1}{\left(m_1 + \frac{1}{2}m_2\right)s^2 + \left(\frac{b_r}{r^2}\right)s + k}$$

Problem 2. **(40%)** Given the equation of motion:

$$\ddot{y} + 9y = u(t),$$

find:
(a) The system's free response provided $u(t) = 0$, $y(0) = 1$, and $\dot{y}(0) = 0$.
(b) The system's forced response provided $u(t) = 1$ (unit step), $y(0) = 0$, and $\dot{y}(0) = 0$.
(c) The system's total response provided $u(t) = 1$ (unit step), $y(0) = 1$, and $\dot{y}(0) = 0$.
(d) The steady-state value of the response associated with part (c).

(a) $\mathcal{L} \Rightarrow s^2 Y(s) - s\overset{1}{\cancel{y(0)}} - \overset{0}{\cancel{\dot{y}(0)}} + 9Y(s) = 0$

$$\Rightarrow Y(s) = \frac{s}{s^2+9}$$

$$\mathcal{L}^{-1}: \; y(t) = \cos 3t$$

(b) $\mathcal{L} \Rightarrow s^2 Y(s) + 9Y(s) = \mathcal{U}(s) = \frac{1}{s}$

$$\Rightarrow Y(s) = \frac{1}{s(s^2+9)}$$

$$= \frac{A}{s} + \frac{Bs+C}{s^2+9}$$

$$\Rightarrow 1 = As^2 + 9A + Bs^2 + Cs$$

$$\Rightarrow 0 = A + B, \; 0 = C, \; 1 = 9A$$

$$\Rightarrow A = \frac{1}{9}, B = -\frac{1}{9}$$

This page is for extra work related to Problem 2.

$$\therefore \; Y(s) = \frac{1}{9}\left(\frac{1}{s}\right) - \frac{1}{9}\left(\frac{s}{s^2+9}\right)$$

$$\mathcal{L}^{-1}: \; y(t) = \frac{1}{9} - \frac{1}{9}\cos 3t$$

(c) $y(t) = y_a + y_b = \cos 3t + \frac{1}{9} - \frac{1}{9}\cos 3t$

$$= \frac{1}{9} + \frac{8}{9}\cos 3t$$

(d) $y_{ss} = y(t) = \frac{1}{9} + \frac{8}{9}\cos 3t$

Problem 3. (20%) Given a single input - single output (SISO) system with poles and zeros:

$$p_{1,2} = \pm 2j, \qquad p_3 = -1, \qquad z_1 = 2,$$

find:
(a) The transfer function of the system.
(b) The differential equation governing the system's ouput $y(t)$ and input $u(t)$
(c) The system's stability (stable, unstable, or marginally stable).

(a) $$G(s) = \frac{s-2}{(s+2j)(s-2j)(s+1)}$$

$$= \frac{s-2}{(s^2+4)(s+1)}$$

$$= \frac{s-2}{s^3+s^2+4s+4}$$

(b) $$G(s) = \frac{Y(s)}{U(s)} = \frac{s-2}{s^3+s^2+4s+4}$$

$$\Rightarrow (s^3+s^2+4s+4)Y(s) = (s-2)U(s)$$

$$\Rightarrow \dddot{y} + \ddot{y} + 4\dot{y} + 4y = \dot{u} - 2u$$

(c) Poles on Imaginary Axis $\Rightarrow$ marginally stable