ME 375 EXAM #1
Tuesday, Oct. 6, 2009

Division: Meckl 10:30 / Deng 2:30 (circle one) **Name** Solution

Instructions

(1) This is a closed book examination, but you are allowed one 8.5×11 crib sheet.

(2) You have one hour to work all three problems on the exam.

(3) Use the solution procedure we have discussed: what are you given, what are you asked to find, what are your assumptions, what is your solution, does your solution make sense. You must show all of your work to receive <u>any</u> credit.

(4) You must write neatly and should use a logical format to solve the problems. You are encouraged to really "think" about the problems before you start to solve them. Please write your name in the top right-hand corner of each page.

(5) A table of Laplace transform pairs and properties of Laplace transforms is attached at the end of this exam set.

Problem No. 1 (40%) ________________

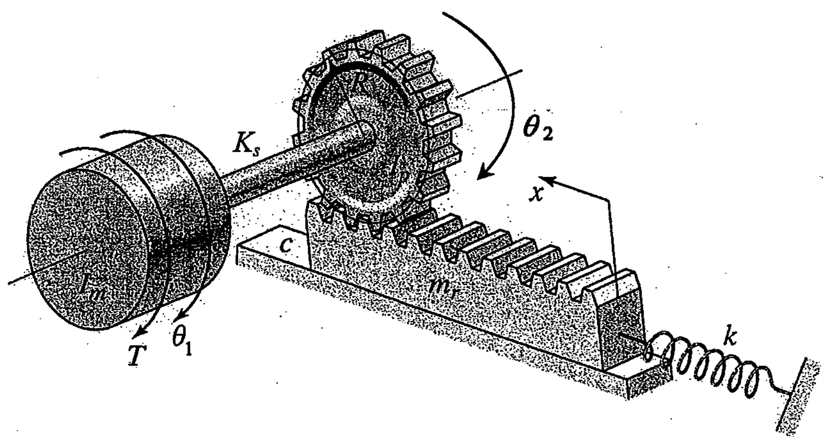
Problem No. 2 (40%) ________________

Problem No. 3 (20%) ________________

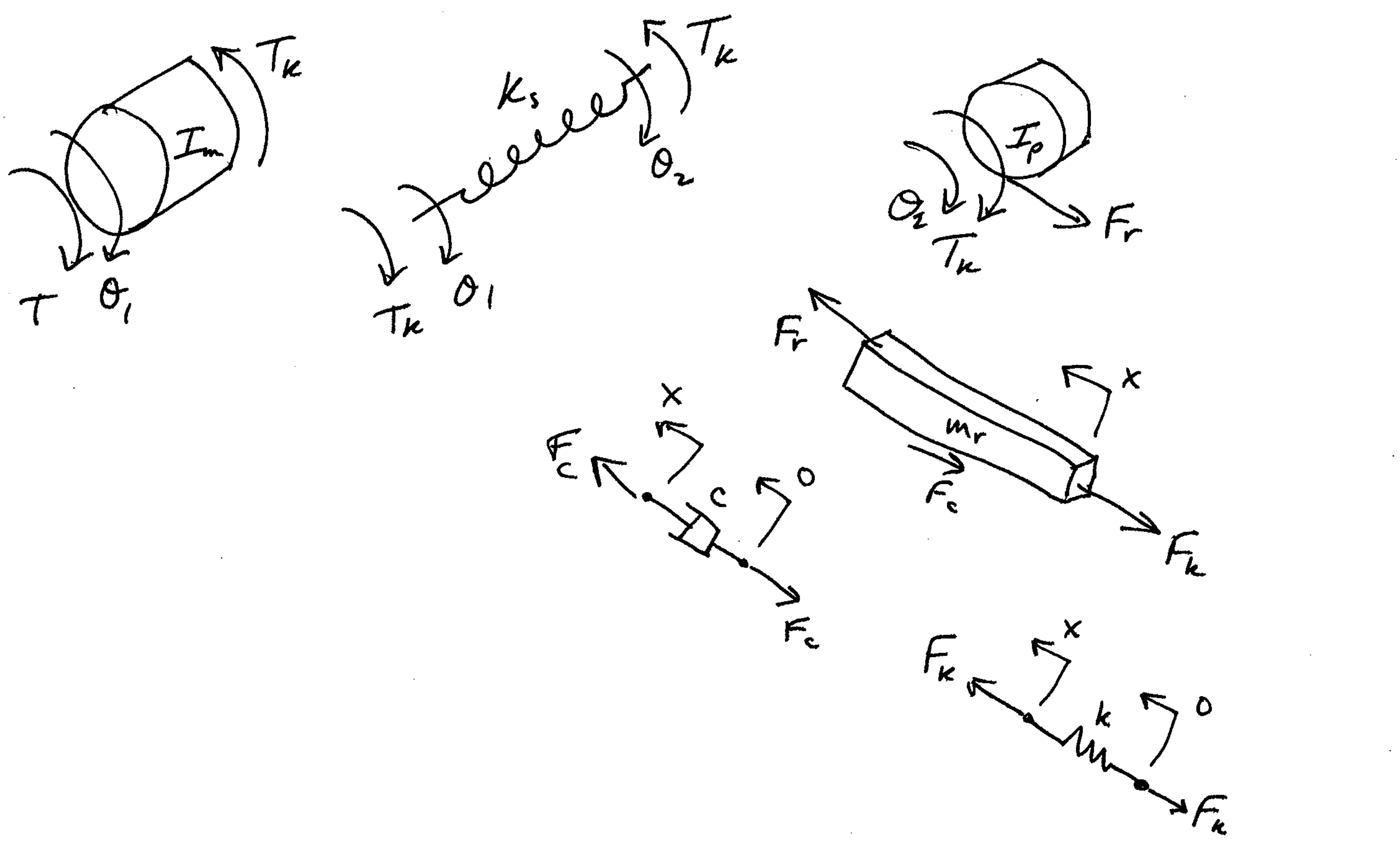
TOTAL (*/100%)** ________________

PROBLEM 1: (40%)

The schematic shown below represents a rack and pinion system. The three coordinates shown are absolute coordinates drawn with respect to the static equilibrium position of the system. Assume there is no slip between the rack and pinion. The motor inertia is I_m, the shaft has stiffness K_s, the pinion gear has inertia I_p and radius R, the rack has mass m_r and experiences viscous damping c with ground.



a) Draw free body diagrams for all the elements in this system. Make sure to include all displacements and label all the forces.



b) Write down the elemental equations.

$$I_m \ddot{\theta}_1 = T - T_k$$

$$T_k = K_s(\theta_1 - \theta_2)$$

$$I_p \ddot{\theta}_2 = T_k - F_r R$$

$$m_r \ddot{x} = F_r - F_c - F_k$$

$$F_c = c\dot{x}$$

$$F_k = kx$$

c) Using $T(t)$ as the input to the system, show that the equations of motion (EOM) are given by:

$$I_m \ddot{\theta}_1 + K_s \theta_1 - \frac{K_s}{R} x = T$$

$$\left(m_r + \frac{I_p}{R^2}\right)\ddot{x} + c\dot{x} + \left(k + \frac{K_s}{R^2}\right)x - \frac{K_s}{R}\theta_1 = 0$$

$$I_m \ddot{\theta}_1 = T - K_s(\theta_1 - \theta_2)$$

$$I_m \ddot{\theta}_1 + K_s \theta_1 - K_s \theta_2 = T \qquad (1)$$

$$m_r \ddot{x} = F_r - c\dot{x} - kx \Rightarrow F_r = m_r \ddot{x} + c\dot{x} + kx$$

$$I_p \ddot{\theta}_2 = T_k - F_r R = K_s(\theta_1 - \theta_2) - R(m_r \ddot{x} + c\dot{x} + kx) \qquad (2)$$

need one more eq. to eliminate θ_2 : $x = R\theta_2 \Rightarrow \theta_2 = \frac{x}{R}$

$$\boxed{I_m \ddot{\theta}_1 + K_s \theta_1 - \frac{K_s}{R} x = T} \qquad (1)'$$

$$\frac{I_p}{R}\ddot{x} + \frac{K_s}{R} x + m_r R \ddot{x} + cR\dot{x} + kRx - K_s \theta_1 = 0$$

$$\boxed{\left(m_r + \frac{I_p}{R^2}\right)\ddot{x} + c\dot{x} + \left(k + \frac{K_s}{R^2}\right)x - \frac{K_s}{R}\theta_1 = 0} \qquad (2)'$$

PROBLEM 2: (40%)

A system has the following model:

$$\ddot{y}+4\dot{y}+5y=5u$$

where y is the output and u is the input of the system.

a) Find the transfer function $G(s)=\dfrac{Y(s)}{U(s)}$.

$$\boxed{G(s)=\frac{Y(s)}{U(s)}=\frac{5}{s^2+4s+5}}$$

b) Find the forced response $y(t)$ with unit step input $u(t)$ and zero initial conditions.

$$Y(s)=\frac{5}{s^2+4s+5}U(s)=\frac{5}{s^2+4s+5}\,\frac{1}{s}$$

$$=\frac{5}{s\left[(s+2)^2+1^2\right]}=\frac{A}{s}+\frac{Bs+C}{(s+2)^2+1^2}=\frac{As^2+4As+5A+Bs^2+Cs}{s\left[(s+2)^2+1^2\right]}$$

s^2: $A+B=0 \Rightarrow B=-1$

s^1: $4A+C=0 \Rightarrow C=-4$

s^0: $5A=5 \Rightarrow A=1$

$$\Bigg\}\quad Y(s)=\frac{1}{s}-\frac{s+4}{(s+2)^2+1^2}$$

$$Y(s)=\frac{1}{s}-\frac{s+2}{(s+2)^2+1^2}-\frac{2(1)}{(s+2)^2+1^2}$$

$\mathcal{Y}^{-1}$

$$\boxed{y(t)=1-e^{-2t}\cos t-2e^{-2t}\sin t}$$

c) Find the free response $y(t)$ with $y(0)=1$ and $\dot{y}(0)=-1$. $\ddot{y}+4\dot{y}+5y=0$

$$\left[s^2Y(s)-sy(0)-\dot{y}(0)\right]+4\left[sY(s)-y(0)\right]+5Y(s)=0$$

$$(s^2+4s+5)\,Y(s)=s+3$$

$$Y(s)=\frac{s+3}{s^2+4s+5}=\frac{s+3}{(s+2)^2+1^2}=\frac{s+2}{(s+2)^2+1^2}+\frac{1}{(s+2)^2+1^2}$$

$\mathcal{L}^{-1}$

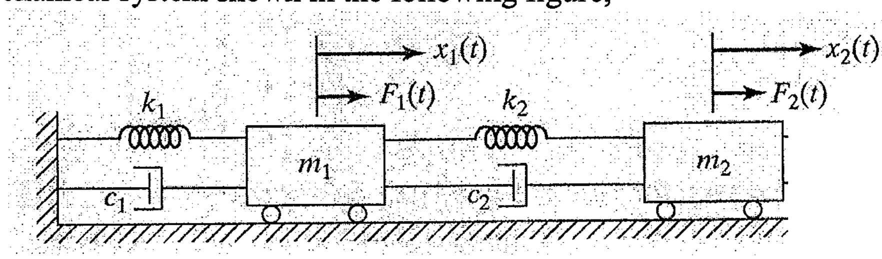
$$\boxed{y(t)=e^{-2t}\cos t+e^{-2t}\sin t}$$

d) Is this system stable? Please explain. Yes

free response decays to zero.

PROBLEM 3: (20%)

Consider the mechanical system shown in the following figure,



The equations of motion for this system are given by

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = F_1$$
$$m_2\ddot{x}_2 + c_2\dot{x}_2 - c_2\dot{x}_1 + k_2x_2 - k_2x_1 = F_2$$

Define an appropriate state vector $q = [q_1\ q_2\ q_3\ q_4]^T$ and write the state space form of this model. Assume the output to be $x_2(t)$.

Let $q_1 = x_1$, $q_2 = \dot{x}_1$, $q_3 = x_2$, $q_4 = \dot{x}_2$

$$\boxed{\dot{q}_1 = \dot{x}_1 = q_2}$$

$$\dot{q}_2 = \ddot{x}_1 = -\frac{k_1+k_2}{m_1}x_1 - \frac{c_1+c_2}{m_1}\dot{x}_1 + \frac{k_2}{m_1}x_2 + \frac{c_2}{m_1}\dot{x}_2 + \frac{1}{m_1}F_1$$

$$\boxed{\begin{aligned}\dot{q}_2 &= -\frac{k_1+k_2}{m_1}q_1 - \frac{c_1+c_2}{m_1}q_2 + \frac{k_2}{m_1}q_3 + \frac{c_2}{m_1}q_4 + \frac{1}{m_1}F_1\\ \dot{q}_3 &= \dot{x}_2 = q_4\end{aligned}}$$

$$\dot{q}_4 = \ddot{x}_2 = \frac{k_2}{m_2}x_1 + \frac{c_2}{m_2}\dot{x}_1 - \frac{k_2}{m_2}x_2 - \frac{c_2}{m_2}\dot{x}_2 + \frac{1}{m_2}F_2$$

$$\boxed{\dot{q}_4 = \frac{k_2}{m_2}q_1 + \frac{c_2}{m_2}q_2 - \frac{k_2}{m_2}q_3 - \frac{c_2}{m_2}q_4 + \frac{1}{m_2}F_2}$$

$$\boxed{y = x_2 = q_3}$$