Name: ________________________________

Section: 8:30_Seipel 12:30_Savran

ME 375

EXAM I

Thursday, 6 October 2011

6:30 pm-7:30 pm

PHYS 114

Problem 1 ________________/35

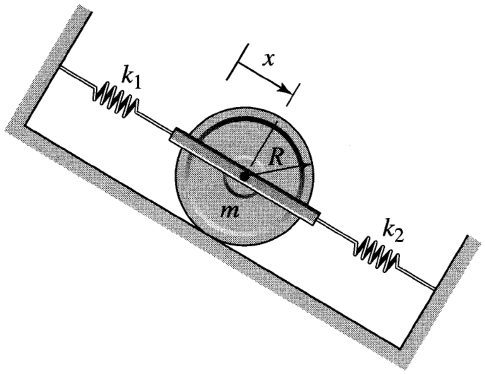
Problem 2________________/30

Problem 3________________/35

Total Score:_____________/100

- Don't forget your name and please circle your section instructor.
- Exam is 5 pages including this cover page: make sure you're not missing any pages.
- **If you use extra pages, indicate on the problem page that you're continuing onto an extra page.**
- Pay attention to units if they are given.
- Show your work. Correct answer with no or wrong explanation = no credit.

Problem 1.
Derive the equation of motion in terms of x. Assume rolling without slipping, x is measured with respect to equilibrium, and the moment of inertia of the wheel about its center is $I = mR^2/2$.



Solution: Summing forces acting on the wheel in the x direction:

$$m\ddot{x} = -k_1 x - k_2 x - T,$$

where T is the force tangent to the ground acting at the point where the wheel contacts ground.

Summing torques about the wheel center of mass, where positive torques correspond with positive wheel rotation θ (assuming that positive θ motion corresponds with positive x direction):

$$mR^2/2\ddot{\theta} = TR.$$

The no-slipping constraint ensures that the translation of the center of the wheel x is related to the positive θ direction by the following:

$$x = R\theta\,,\ \ \ddot{x} = R\ddot{\theta}.$$

Combining these equations we find:

$$m\ddot{x} = -k_1 x - k_2 x - T,$$

$$m\ddot{x} = -k_1 x - k_2 x - mR/2\ddot{\theta},$$

$$m\ddot{x} = -k_1 x - k_2 x - m/2\ddot{x},$$

$$m\ddot{x} + m/2\ddot{x} = -k_1 x - k_2 x,$$

and finally:

$$1.5m\ddot{x} = -(k_1 + k_2)x.$$

Alternatively, the solution can be found by summing the moments about the instantaneous center of rotation (the contact point between wheel and ground), and converting the equation in terms of x using the kinematic relationship:

$$(mR^2/2 + mR^2)\ddot{\theta} = -k_1 x R - k_2 x R,$$
$$(mR/2 + mR)\ddot{x} = -k_1 x R - k_2 x R,$$
$$1.5m\ddot{x} = -(k_1 + k_2)x.$$

Problem 2. (part a is a separate question on its own. it is **not** related to the other parts).

a) Determine the Laplace transform, X(s), of the following expression:

$$x(t) = 1 + 3t^3 + 2e^{-2t} + 4te^{-t} + 5e^{-t}\cos 5t$$

$$X(s) = \frac{1}{s} + \frac{3 \times 3 \times 2}{s^4} + 2 \cdot \frac{1}{s+2} + 4 \frac{1}{(s+1)^2} + 5 \frac{s+1}{(s+1)^2 + 25}$$

$$= \frac{1}{s} + \frac{18}{s^4} + \frac{2}{s+2} + \frac{4}{(s+1)^2} + 5\frac{(s+1)}{(s+1)^2+25}$$

b) What is the **final value**, $y(\infty)$, of $Y(s) = \dfrac{s+6}{s(s^2+3s+6)}$?

Answer the question <u>without</u> calculating the inverse Laplace of Y(s).

Final value theorem: $\lim_{s \to 0} \cancel{s} \dfrac{s+6}{\cancel{s}(s^2+3s+6)} = \dfrac{6}{6} = 1$

c) Someone tells you that the above Y(s) is the forced response of a system G(s), when it's excited by a unit step. Then, what is G(s)?

$$Y_{\text{forced}}(s) = G(s)\underbrace{U(s)}_{\frac{1}{s}} = \frac{s+6}{s(s^2+3s+6)} \Rightarrow G(s) = \frac{s+6}{s^2+3s+6}$$

d) Based on G(s) you found in part c, what is the differential equation that relates the output $y(t)$ to an input $u(t)$?

$$\ddot{y} + 3\dot{y} + 6y = \dot{u} + 6u$$

Problem 3.

A system is described by the following ODE:

$$\ddot{y}(t)+2\dot{y}(t)+4y(t)=\dot{u}(t)+u(t)$$

where $u(t)$ is the input and $y(t)$ is the output.

a) What are the poles of this system?

$G(s) = \dfrac{s+1}{s^2+2s+4}$ roots of $s^2+2s+4=0$: $\boxed{s_{1,2}=-1 \pm 1.73j}$

b) What are the zeros of the system?

root of $s+1=0$ $\boxed{s=-1}$

c) Discuss the stability of the system.

system is stable because real part of poles is negative.

Next, we'll determine the total response of this system to a unit step $u(t)=1$, with the initial conditions: $y(0)=0$ and $\dot{y}(0)=1$.

d) First obtain Y(s) and express it as the sum of the forced and the free responses. Indicate which part is forced and which part is free.

$$s^2Y(s) - sy(0) - \dot{y}(0) + 2sY(s) - 2y(0) + 4Y(s) = sU(s) + U(s)$$

$$Y(s)(s^2+2s+4) - sy(0) - \dot{y}(0) - 2y(0) = sU(s) + U(s)$$

$$Y(s) = \underbrace{\frac{s+1}{s^2+2s+4}U(s)}_{\text{FORCED}} + \underbrace{\frac{sy(0)+\dot{y}(0)+2y(0)}{s^2+2s+4}}_{\text{FREE}}$$

b) Now obtain y(t).

$$Y(s) = \frac{s+1}{s(s^2+2s+4)} + \frac{1}{s^2+2s+4} = \frac{2s+1}{s(s^2+2s+4)}$$

$2^2 - 4.4 = 4-16 < 0 \Rightarrow$ imaginary poles

$$Y(s) = \frac{A}{s} + \frac{B(s+1)}{(s+1)^2+3} + \frac{C\sqrt{3}}{(s+1)^2+3}$$

$$A(s^2+2s+4) + Bs(s+1) + Cs\sqrt{3} = 2s+1$$

$$As^2 + 2As + 4A + Bs^2 + Bs + \sqrt{3}Cs = 2s+1$$

$$(A+B)s^2 + (2A+B+\sqrt{3}C)s + 4A = 2s+1$$

$$\Rightarrow A = \frac{1}{4} \qquad A+B=0 \Rightarrow B = -\frac{1}{4}$$

$$2A+B+\sqrt{3}C = 2 \Rightarrow \frac{1}{2} - \frac{1}{4} + \sqrt{3}C = 2 \Rightarrow C = 1.01$$

$$y(t) = \frac{1}{4} - \frac{1}{4}e^{-t}\cos\sqrt{3}t + 1.01\,e^{-t}\sin\sqrt{3}t$$