ME 375 EXAM #1
Tuesday, Sep. 25th, 2012

Division Hall / Deng (circle one) **Name** Solution

Instructions

(1) This is a closed book examination, but you are allowed one 8.5×11 crib sheet.

(2) You have one hour to work all three problems on the exam.

(3) Use the solution procedure we have discussed: what are you given, what are you asked to find, what are your assumptions, what is your solution, does your solution make sense. You must show all of your work to receive <u>any</u> credit.

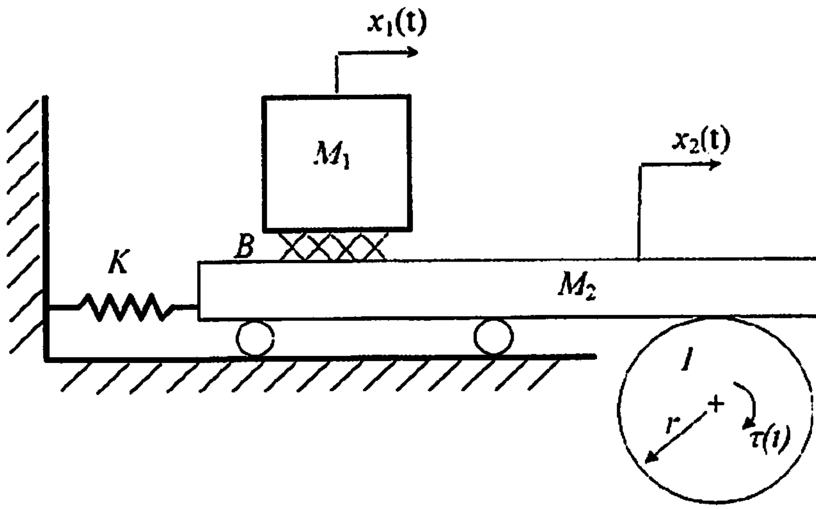
(4) You must write neatly and should use a logical format to solve the problems. You are encouraged to really "think" about the problems before you start to solve them. Please write your name in the top right-hand corner of each page.

(5) A table of Laplace transform pairs and properties of Laplace transforms is located on the last page of the exam.

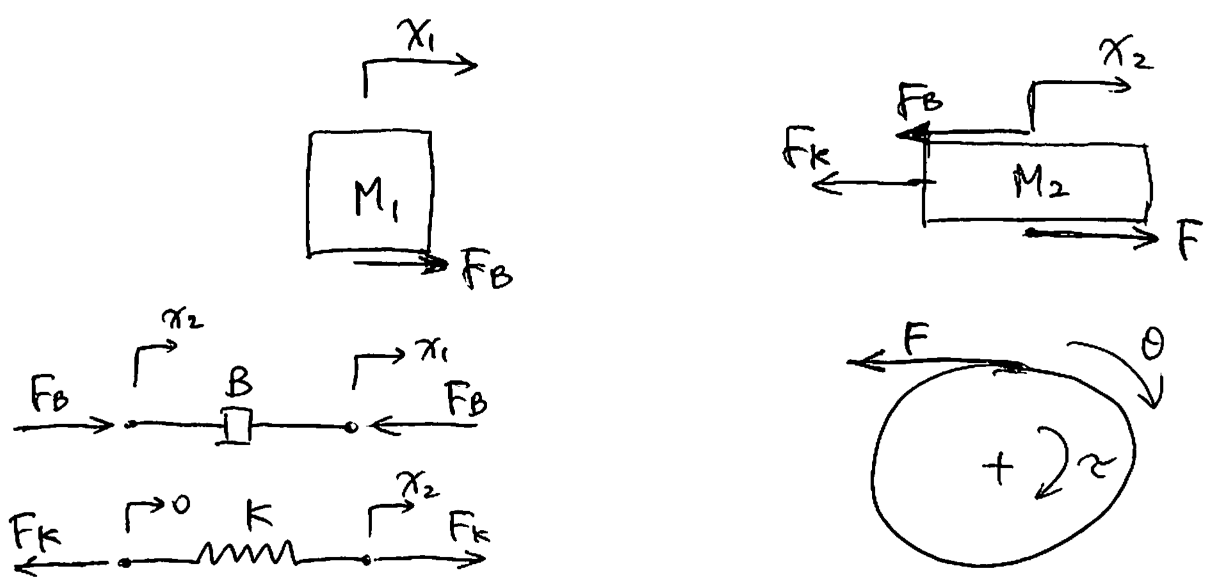
Problem No. 1 (40%)	__________
Problem No. 2 (30%)	__________
Problem No. 3 (30%)	__________
TOTAL (*/100%)**	__________

PROBLEM 1: (40%)

Consider the following system, where $\tau(t)$ is an external input torque to the system. Assume that the center of the disk is fixed (so that there is not translation of the disk), and the disk rolls without slipping on the mass M_2. I and r are the moment of inertia and radius of the disk, respectively. K and B are the spring and damping constant, respectively.



(a) Draw free body diagrams for all the elements in this system. Make sure to include all displacements and label all the forces in the coordinates of interest.



PROBLEM 1: (continued)

(b) Write down the element equations.

mass M_1: $M_1 \ddot{x}_1 = F_B$ (1)

$F_B = B(\dot{x}_2 - \dot{x}_1)$ (2)

mass M_2: $M_2 \ddot{x}_2 = F - F_K - F_B$ (3)

$F_K = K x_2$ (4)

disk I: $I \ddot{\theta} = \tau - rF$ (5)

Coupling: $x_2 = r\theta$ (6)

(c) Derive the equations of motion for this system.

From (1) & (2):

$$M_1 \ddot{x}_1 = B(\dot{x}_2 - \dot{x}_1) \Rightarrow \boxed{M_1 \ddot{x}_1 + B\dot{x}_1 - B\dot{x}_2 = 0} \quad (7)$$

From (5) & (6):

$$F = \frac{1}{r}(\tau - I\ddot{\theta}) = \frac{1}{r}\left(\tau - \frac{I}{r}\ddot{x}_2\right) \quad (8)$$

Substitute (8) & (2) & (4) into (3):

$$M_2 \ddot{x}_2 = \frac{1}{r}\left(\tau - \frac{I}{r}\ddot{x}_2\right) - Kx_2 - B(\dot{x}_2 - \dot{x}_1)$$

i.e. $$\boxed{\left(M_2 + \frac{I}{r^2}\right)\ddot{x}_2 + B\dot{x}_2 + Kx_2 - B\dot{x}_1 = \frac{1}{r}\tau} \quad (9)$$

PROBLEM 2: (30%)

Given the following equations of motion of a mechanical system,

$$M_1\ddot{x}_1 + (B_1 + B_2)\dot{x}_1 + K_1 x_1 - B_2\dot{x}_2 = 0 \qquad (1)$$
$$M_2\ddot{x}_2 - B_2\dot{x}_1 + B_2\dot{x}_2 = f(t) \qquad (2)$$

(a) Write the Laplace Transforms of the equations of motion (assuming zero initial conditions).

$$[M_1 s^2 + (B_1 + B_2)s + K_1] X_1(s) - B_2 s X_2(s) = 0 \qquad (1)$$

$$[M_2 s^2 + B_2 s] X_2(s) - B_2 s X_1(s) = F(s) \qquad (2)$$

(b) Find the transfer function $G(s) = \dfrac{X_2(s)}{F(s)}$ where $F(s)$ is the input and $X_2(s)$ is the output.

From (2): $$X_1(s) = \frac{[M_2 s^2 + B_2 s] X_2(s) - F(s)}{B_2 s} \qquad (3)$$

Substitute (3) into (1), we can get:

$$X_2(s) = \frac{[M_1 s^2 + (B_1 + B_2)s + K_1] \quad F(s)}{[M_1 s^2 + (B_1 + B_2)s + K_1][M_2 s^2 + B_2 s] - (B_2 s)^2}$$

Therefore,

$$G(s) = \frac{X_2(s)}{F(s)} = \frac{[M_1 s^2 + (B_1 + B_2)s + K_1]}{[M_1 s^2 + (B_1 + B_2)s + K_1][M_2 s^2 + B_2 s] - (B_2 s)^2}$$

PROBLEM 2: (continued)

$$G(s) = \frac{X_2(s)}{F(s)} = \frac{[M_1 s^2 + (B_1+B_2)s + K_1]}{M_1M_2 s^4 + [M_1B_2 + M_2(B_1+B_2)]s^3 + (K_1M_2 + B_1B_2)s^2 + K_1B_2 s}$$

(c) Find the input/output differential equation where $f(t)$ is the input and $x_2(t)$ is the output.

$$M_1M_2\ddddot{x}_2 + [M_1B_2 + M_2(B_1+B_2)]\dddot{x}_2 + (K_1M_2 + B_1B_2)\ddot{x}_2 + K_1B_2\dot{x}_2$$

$$= M_1\ddot{f} + (B_1+B_2)\dot{f} + K_1 f$$

PROBLEM 3: (30%)

Given the following differential equation,

$$\ddot{y}+6\dot{y}+10y=10u(t)$$

(a) Find the system's transfer function $G(s)=\dfrac{Y(s)}{U(s)}$.

$$G(s) = \frac{10}{s^2+6s+10}$$

(b) Find the forced response $y(t)$ to a unit step input $u(t)$.

$$Y(s) = G(s)\cdot U(s)$$

$$= \frac{10}{s(s^2+6s+10)} = \frac{A}{s} + \frac{Bs+C}{s^2+6s+10}$$

$$A(s^2+6s+10) + Bs^2 + Cs = 10 \Rightarrow \begin{cases} A=1 \\ B=-1 \\ C=-6 \end{cases}$$

$$Y(s) = \frac{1}{s} - \frac{s+6}{s^2+6s+10} = \frac{1}{s} - \frac{s+6}{(s+3)^2+1^2}$$

$$= \frac{1}{s} - \frac{s+3}{(s+3)^2+1^2} - 3\cdot\frac{1}{(s+3)^2+1^2}$$

$$\therefore y(t) = 1 - e^{-3t}\cos t - 3e^{-3t}\sin t$$

PROBLEM 3: (continued)

(c) Find the free response $y(t)$ with $y(0) = 1$ and $\dot{y}(0) = -1$.

$$[s^2 Y(s) - s y(0) - \dot{y}(0)] + 6[sY(s) - y(0)] + 10Y(s) = 0$$

$$(s^2 + 6s + 10)Y(s) = sy(0) + \dot{y}(0) + 6y(0)$$

$$= s + 5$$

$$Y(s) = \frac{s+5}{s^2+6s+10}$$

$$= \frac{s+5}{(s+3)^2+1^2}$$

$$= \frac{s+3}{(s+3)^2+1^2} + 2 \cdot \frac{1}{(s+3)^2+1^2}$$

$$\therefore\ y(t) = e^{-3t}\cos t + 2e^{-3t}\sin t$$

(d) Is this system stable? Please explain.

Yes, the poles of the transfer function are at $s = -3 \pm j$, which is on the LHP of s-plane. (i.e. negative real parts).