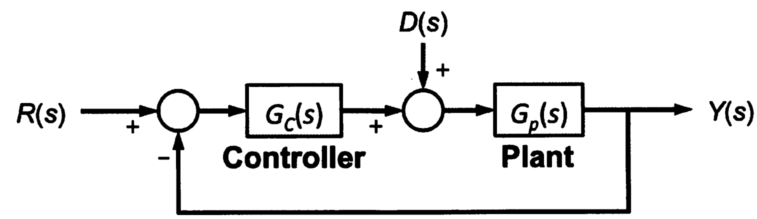
Name: ____________________ HWID: ____________________

PROBLEM 1 (30 points)

Consider the following feedback system, where $R(s)$ is a reference input and $D(s)$ is a disturbance input:



For this system, we have

$$G_p(s) = \frac{1}{s(10s+3)}$$

(A) (15 points) If we have a PD controller in the form of:

$$G_c(s) = K_P + K_D s$$

The performance specification require the steady state error due to a unit step reference input ($R(s) = \frac{1}{s}$) to be zero and the steady-state error due to a unit step disturbance input ($D(s) = \frac{1}{s}$) to be no greater than 0.1 in magnitude. What values of gains K_P and K_D will satisfy these conditions?

system is of type 1 for reference input, $e_{ss}\big|_{\text{step reference}} = 0$;

$\forall K_p > 0$

$\forall K_D > 0$

$$\frac{E(s)}{D(s)} = \frac{-G_p}{1+G_c G_p} = \frac{-\frac{1}{s(10s+3)}}{1+\frac{K_p+K_D s}{s(10s+3)}}$$

$$= -\frac{1}{s(10s+3)+K_p+K_D s}$$

$$e_{ss}\Big|_{\text{step disturbance}} = \lim_{s\to 0} s \cdot \frac{E(s)}{D(s)} \cdot \underbrace{\frac{1}{s}}_{D(s)} = \lim_{s\to 0} -\frac{1}{s(10s+3)+K_p+K_D s}$$

$$= -\frac{1}{K_p}$$

$$\left|-\frac{1}{K_p}\right| \le 0.1 \Rightarrow K_p \ge 10. \quad K_D > 0.$$

PROBLEM 1 (continued)

(B) (15 points) If we would like to use pole placement to design a controller in the form of:

$$G_c(s)=\frac{as+b}{cs+d}$$

Determine the coefficients a, b, c, and d using the direct pole placement method to locate the roots such that the desired closed-loop characteristic polynomial is $(s+5)(s^2+4s+10)=s^3+9s^2+30s+50$.

$$D_p D_c + N_p N_c = D_{cl}$$

$$s(10s+3)(cs+d)+(as+b) = \underbrace{s^3+9s^2+30s+50}$$

$$10cs^3+(3c+10d)s^2+(3d+a)s+b = \ldots\downarrow\ldots$$

$$10c = 1 \Rightarrow c = 0.1$$

$$3c+10d = 9 \Rightarrow d = \frac{9-3\times 0.1}{10} = 0.87$$

$$3d+a = 30 \Rightarrow a = 30-3\times 0.87 = 27.39$$

$$b = 50$$

$$\therefore\ G_c = \frac{27.39s+50}{0.1s+0.87}$$

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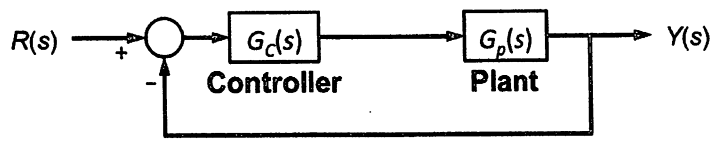
PROBLEM 2 (30 points)

Consider the following unity feedback system:



where

$$G_c(s)=K \text{ and } G_p(s)=\frac{1}{s^3+5s^2+20s+10}$$

(A) (15 points) Using Routh stability test, find the range of K such that the closed loop system is stable.

$$TF: \frac{G_c G_p}{1+G_c G_p} = \frac{\frac{K}{s^3+5s^2+20s+10}}{1+\frac{K}{s^3+5s^2+20s+10}}$$

$$= \frac{K}{s^3+5s^2+20s+10+K}$$

s^3	1	20
s^2	5	$10+K$
s^1	b_1	0
s^0	c_1	

$$b_1 = \frac{\begin{vmatrix} 1 & 20 \\ 5 & 10+K \end{vmatrix}}{-5} = \frac{90-K}{5}$$

$$c_1 = \frac{\begin{vmatrix} 5 & 10+K \\ b_1 & 0 \end{vmatrix}}{-b_1} = 10+K$$

$b_1 > 0 \Rightarrow K < 90$

$c_1 > 0 \Rightarrow K > -10$

$\because K > 0 \therefore 0 < K < 90$

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(B) (15 points) For [redacted] a unit step input R(s), find the sensitivity function of the steady-state error e_{ss} to changes in the control parameter K. i.e., find $S_K^{e_{ss}}$.

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G_c G_p}$$

$$= \frac{1}{1 + \frac{K}{s^3 + 5s^2 + 20s + 10}}$$

$$= \frac{s^3 + 5s^2 + 20s + 10}{s^3 + 5s^2 + 20s + 10 + K}$$

$$e_{ss} = \lim_{s \to 0} s \cdot \frac{E(s)}{R(s)} \cdot \underbrace{\frac{1}{s}}_{}$$

R(s) here.

$$= \frac{10}{10 + K}$$

$$S_K^{e_{ss}} = \frac{K}{e_{ss}} \cdot \frac{\partial e_{ss}}{\partial K} = \frac{K}{\frac{10}{10+K}} \cdot \frac{-10}{(10+K)^2}$$

$$= -\frac{K}{10+K}$$

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PROBLEM 3 (40 points)

For the feedback system shown in Figure 1:

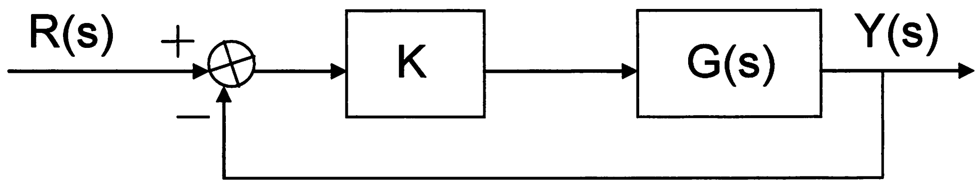

Figure 1. Feedback System

(A) (20 points) For $G(s) = \dfrac{1}{s(s+2)(s+4)}$, The root locus for $K \in [0, \infty)$ is sketched in Figure 2. Find the value of K such that the closed loop system has the smallest settling time.

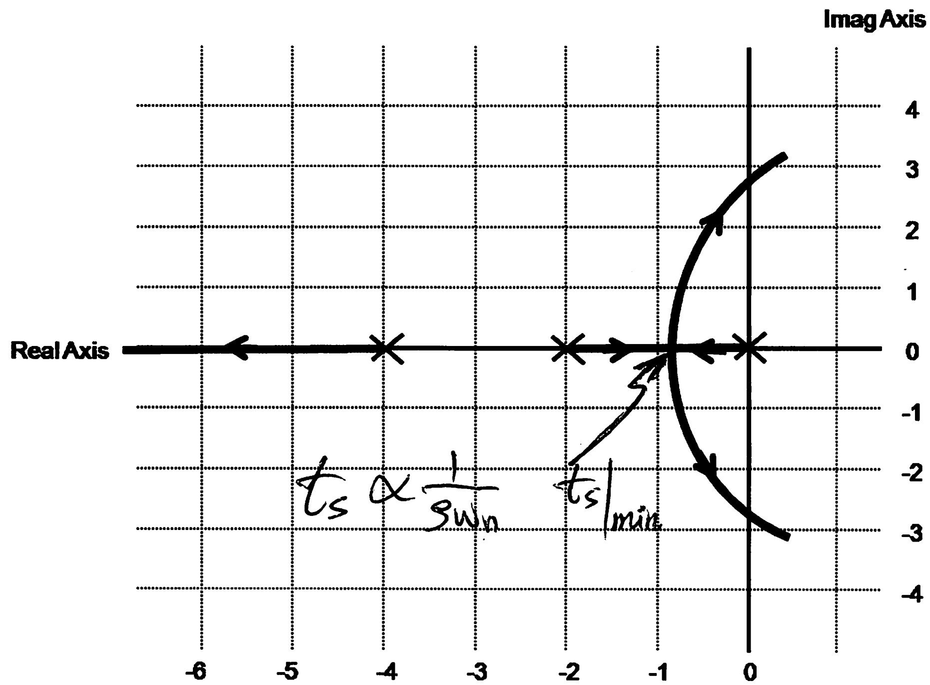

Figure 2. Root locus

break away point has 2 repeated poles, where the slower pole reaches its max|Re| (maximum value of |Re|);

$$1 + K \cdot \frac{1}{s(s+2)(s+4)} = 0$$

$$K = -s(s+2)(s+4)$$
$$= -(s^3 + 6s^2 + 8s)$$

$$\frac{dK}{ds} = -(3s^2 + 12s + 8) = 0$$

$$s = \frac{-12 \pm \sqrt{144-96}}{6} = -0.845 \quad \checkmark$$

$$\text{or } -3.15 \quad \times$$

$$K = -(s^3 + 6s^2 + 8s) = \cancel{11.643}$$

$$3.078.$$

(B) (20 points) If the root locus for the system in Figure 1 is sketched in Figure 3 for $K \in [0, \infty)$,

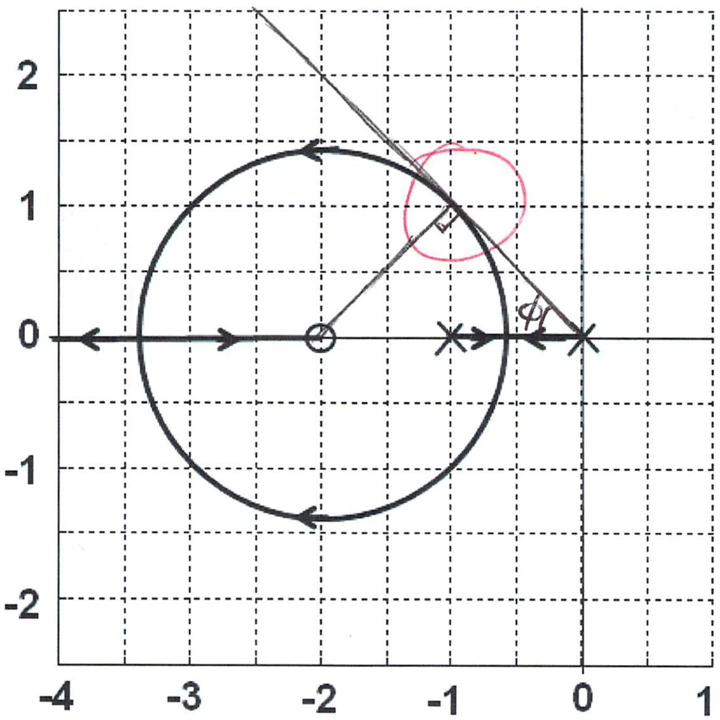

Figure 3. Root locus

(B.1) (10 points) find G(s)

$$G(s) = \frac{s+2}{s(s+1)}$$

(B.2) (10 points) find the smallest damping ratio the system can have for any value of gain K ($K \in [0, \infty)$).

$$\cos\phi = \zeta \Rightarrow \zeta = \cos^{-1} 45^\circ = 0.707$$