ME 375 EXAM #1
Thursday, February 24, 2005

Division King 11:30 / Cunningham 2:30 (circle one) **Name** SOLUTION

Instructions

(1) This is a closed book examination, but you are allowed one 8.5×11 crib sheet.

(2) You have one hour to work all four problems on the exam.

(3) Use the solution procedure we have discussed: what are you given, what are you asked to find, what are your assumptions, what is your solution, does your solution make sense. You must show all of your work to receive <u>any</u> credit.

(4) Circle or box-in your answers.

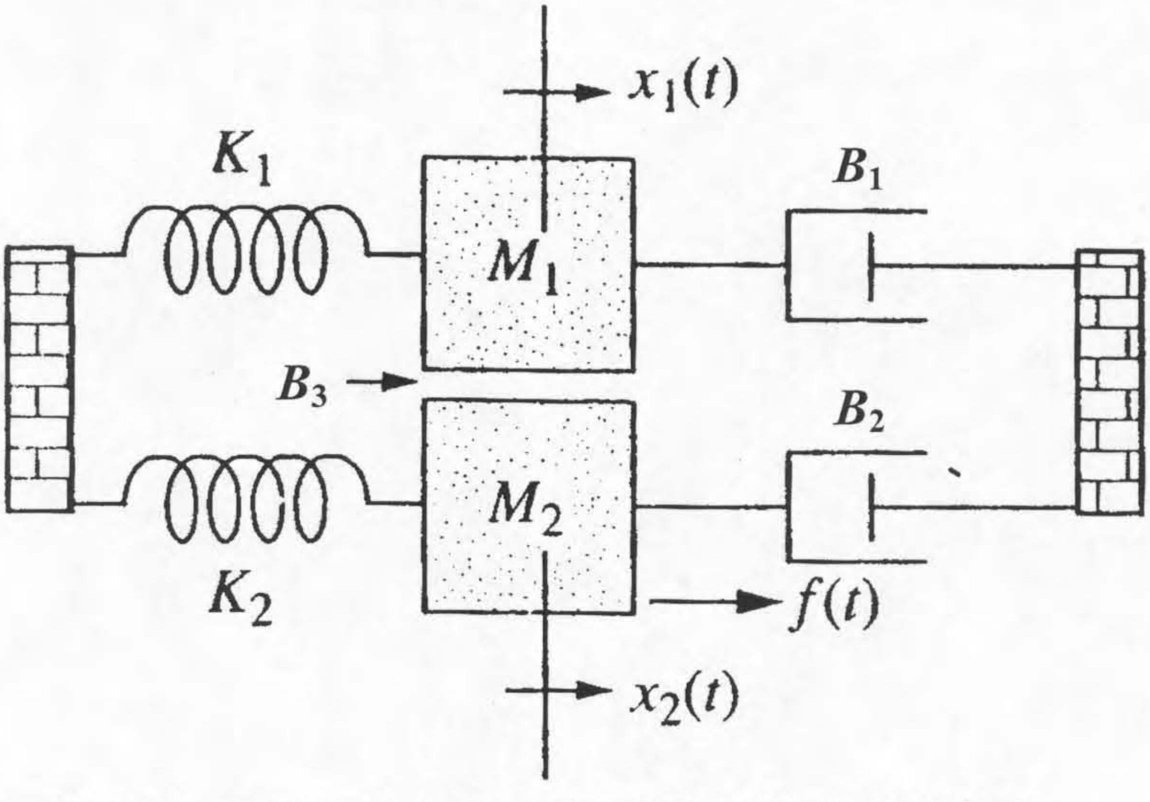
(5) You must write neatly and should use a logical format to solve the problems. You are encouraged to really "think" about the problems before you start to solve them. Please write your name in the top right-hand corner of each page.

(6) A table of Laplace transform pairs and properties of Laplace transforms is attached at the end of this exam set.

Problem No. 1 (30%)	__________
Problem No. 2 (20%)	__________
Problem No. 3 (20%)	__________
Problem No. 4 (30%)	__________
TOTAL (*/100%)**	__________

PROBLEM NO. 1 (30%)

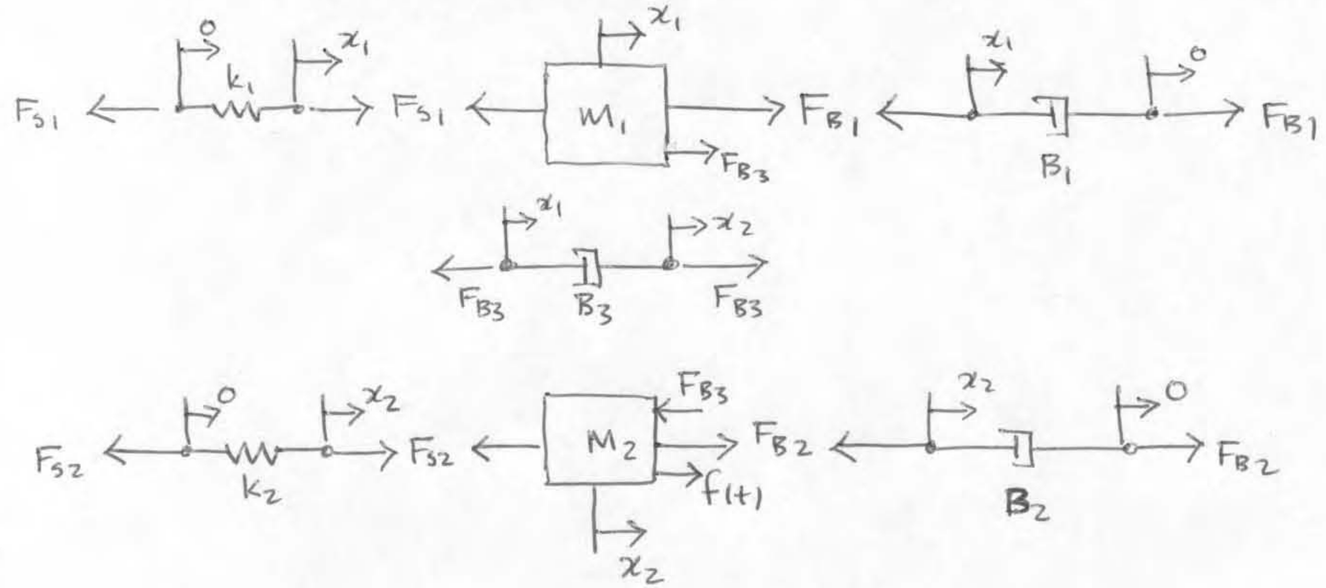
Consider the system shown in the figure below.



(Note: B_3 is the viscous friction between M_1 and M_2. Model this friction as a dashpot)

Develop a model for this system using the following steps:

(a) Draw free body diagrams for all the components of the system shown above. (You can ignore all vertical forces.)



(b) Write governing equations for each of the free body diagrams in part (a). Don't forget to include kinematic relationships if needed.

$$F_{S1} = (k_1 x_1 - 0) = k_1 x_1 \qquad F_{B1} = B_1(0 - \dot{x}_1) = -B_1 \dot{x}_1$$

$$F_{S2} = (k_2 x_2 - 0) = k_2 x_2 \qquad F_{B2} = B_2(0 - \dot{x}_2) = -B_2 \dot{x}_2$$

$$F_{B3} = B_3(\dot{x}_2 - \dot{x}_1)$$

$$M_1 \ddot{x}_1 = F_{B1} + F_{B3} - F_{S1}$$

$$M_2 \ddot{x}_2 = f(t) + F_{B2} - F_{B3} - F_{S2}$$

(c) Develop and write the equations of motion for the system.

$$M_1 \ddot{x}_1 + (B_1 + B_3)\dot{x}_1 + k_1 x_1 - B_3 \dot{x}_2 = 0$$

$$M_2 \ddot{x}_2 + (B_2 + B_3)\dot{x}_2 + k_2 x_2 - B_3 \dot{x}_1 = f(t)$$

PROBLEM NO. 2 **(20%)**

The equations of motion of a mechanical system are as follows:

$$(M_1 s^2 + [B_1 + B_3]s + K_1)X_1(s) - B_3 s X_2(s) = 0 \quad (1)$$

$$-B_3 s X_1(s) + (M_2 s^2 + [B_2 + B_3]s + K_2)X_2(s) = F(s) \quad (2)$$

(a) Find the input/output model of this system where F(s) is the input and $X_1(s)$ is the output.

$$\begin{bmatrix} M_1 s^2 + (B_1+B_3)s + K_1 & -B_3 s \\ -B_3 s & M_2 s^2 + (B_2+B_3)s + K_2 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ F(s) \end{bmatrix}$$

$$X_1(s) = \frac{\begin{vmatrix} 0 & -B_3 s \\ F(s) & M_2 s^2 + (B_2+B_3)s + K_2 \end{vmatrix}}{\begin{vmatrix} M_1 s^2 + (B_1+B_3)s + K_1 & -B_3 s \\ -B_3 s & M_2 s^2 + (B_2+B_3)s + K_2 \end{vmatrix}}$$

$$\frac{X_1(s)}{F(s)} = \frac{B_3 s}{\begin{gathered} M_1 M_2 s^4 + (M_1 B_3 + M_2 B_3 + M_1 B_2 + M_2 B_1) s^3 \\ + (K_1 M_2 + K_2 M_1 + B_3 B_2 + B_3 B_1 + B_2 B_1) s^2 \\ + (K_1 B_3 + K_2 B_3 + K_1 B_2 + K_2 B_1) s + K_1 K_2 \end{gathered}}$$

(b) Find the input/output model of the system where F(s) is the input and the speed (velocity) of X_1 is the output.

$$\frac{V_1(s)}{F(s)} = \frac{s X_1(s)}{F(s)} = \frac{B_3 s^2}{[\qquad\qquad]}$$

PROBLEM No. 3: (20%)

A second-order system with input $f(t)$ is given by

$\ddot{y} + 8\dot{y} + 65y = 3\dot{f} + 26f$,

where $f(t) = 5 \cdot u(t)$, that is, five times the unit step, and the initial conditions are $y(0) = 5$ and $\dot{y}(0) = 1$. Remember to let $f(0) = 0$.

(a) Using Laplace Transforms, solve for the total response for $y(t)$ and identify the forced-response and free-response portions of $y(t)$.

$\mathcal{L}$

$$\left[s^2 Y(s) - s\,\overset{5}{y(0)} - \overset{1}{\dot{y}(0)}\right] + 8\left[sY(s) - \overset{5}{y(0)}\right] + 65\,Y(s) = (3s+26)F(s)$$

$$\left[s^2 + 8s + 65\right] Y(s) = (3s+26)\underset{5/s}{F(s)} + (5s + 41)$$

$$Y(s) = \underbrace{\frac{3s+26}{s^2+8s+65}\left(\frac{5}{s}\right)}_{\text{FORCED}} + \underbrace{\frac{5s+41}{s^2+8s+65}}_{\text{FREE}} = \underbrace{\frac{15s+130}{s\left[(s+4)^2+7^2\right]}}_{\text{FORCED}} + \underbrace{\frac{5s+41}{(s+4)^2+7^2}}_{\text{FREE}}$$

FORCED:

$$Y_{f_{\text{forced}}(s)} = \frac{A}{s} + \frac{B(s+4)}{(s+4)^2+7^2} + \frac{C7}{(s+4)^2+7^2} = \frac{A(s^2+8s+65) + Bs(s+4) + C7s}{s\left[(s+4)^2+7^2\right]}$$

s^2: $A + B = 0 \Rightarrow B = -2$

s^1: $8A + 4B + 7C = 15 \Rightarrow C = \frac{1}{7}(15 + 8 - 16) = 1$

s^0: $65A = 130 \Rightarrow A = 2$

FREE:

$$Y_{\text{free}(s)} = \frac{5s+41}{(s+4)^2+7^2} = \frac{A(s+4)}{(s+4)^2+7^2} + \frac{B\cdot 7}{(s+4)^2+7^2}$$

s^1: $A = 5$

s^0: $4A + 7B = 41 \Rightarrow B = \frac{1}{7}(41-20) = \frac{21}{7} = 3$

$\mathcal{L}^{-1}$

$$\boxed{y(t) = \underbrace{\left[2 - 2e^{-4t}\cos(7t) + e^{-4t}\sin(7t)\right]u(t)}_{\text{FORCED}} + \underbrace{\left[5e^{-4t}\cos(7t) + 3e^{-4t}\sin(7t)\right]u(t)}_{\text{FREE}}}$$

(b) What is $\lim_{t\to\infty} y(t)$? APPLY FVT

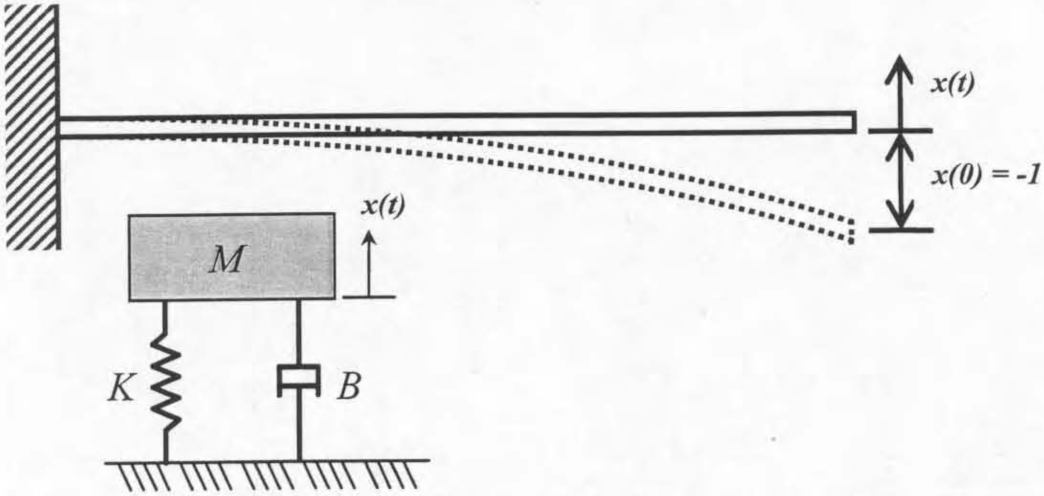
$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \cancel{s}\left(\frac{15s+130}{s^2+8s+65}\right)\left(\frac{1}{\cancel{s}}\right) = \frac{130}{65} = 2$$

$$\boxed{\lim_{t\to\infty} y(t) = 2}$$

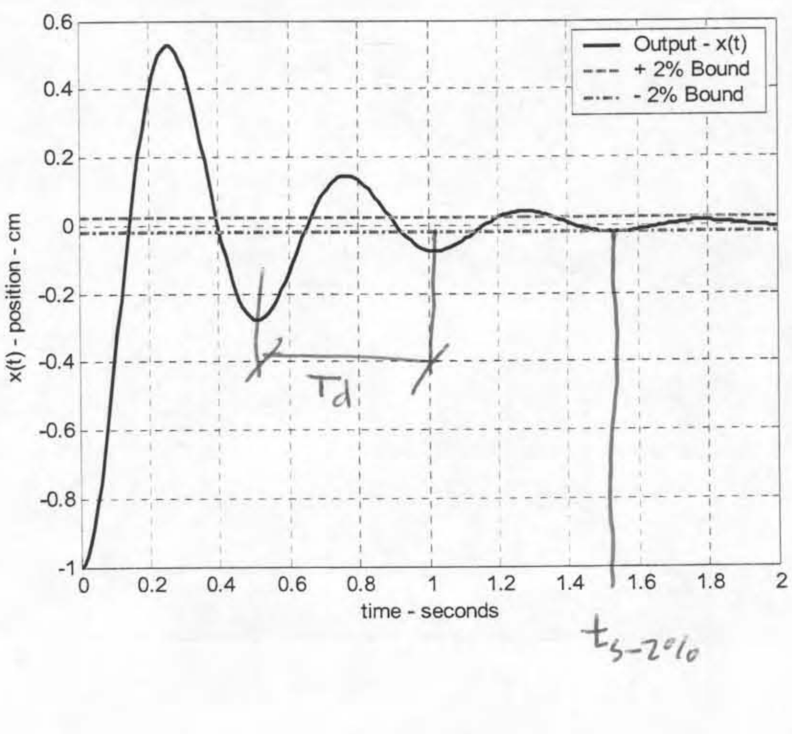
ONLY NEED TO CONSIDER FORCED RESPONSE B/C FREE RESPONSE DIES OUT FOR STABLE SYSTEMS.

PROBLEM No. 4: (30%)

In ME 365, you modeled the cantilever beam in the figure below as an under-damped mass-spring-damper system, also depicted below.



At $t = 0$, the end of the bean is displaced such that $x(0) = -1$ as shown by the dashed lines in the Figure and then released. An example of the "snap-back" response may look like the figure below.



Using this second-order model for the beam and your knowledge of second-order dynamic responses (for example, the period of the damped oscillations, T_d, and the 2% settling time, $t_{s\text{-}2\%}$), how would you calculate the damping ratio, ζ, from the "snap-back" response, where $x(0) = -1$ and $\dot{x}(0) = 0$?

Note that the %OS equation **does not** apply here since it was derived for a step response and this is a "snap-back" or free response.

In your answer:

(1.) explicitly state the steps you would take and

(2.) solve the equations **symbolically** for the damping ratio, ζ, in terms of the parameters you use to describe the second-order "snap-back" response in the plot above.

Specifically mark points and parameters on the example "snap-back" response in the plot above.

(1.) Identify and measure T_d and $t_{s\text{-}2\%}$ from the graph of the "snap-back" response.

(2.) Use $T_d = \dfrac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$ and $t_{s\text{-}2\%} = \dfrac{4}{\zeta\omega_n}$ to solve for ω_n.

(i) from $t_{s\text{-}2\%} \Rightarrow \zeta = \dfrac{4}{t_{s\text{-}2\%}\,\omega_n}$

(ii.) subbing into $T_d \Rightarrow \omega_n^2\left(1 - \dfrac{16}{t_{s\text{-}2\%}^2\,\omega_n^2}\right) = \left(\dfrac{2\pi}{T_d}\right)^2$

$$\omega_n^2 = \frac{16}{t_{s\text{-}2\%}^2} + \left(\frac{2\pi}{T_d}\right)^2$$

$$\omega_n = \sqrt{\frac{16T_d^2 + (2\pi)^2 t_{s\text{-}2\%}^2}{t_{s\text{-}2\%}^2\,T_d^2}}$$

(3.) Use this expression for ω_n to calculate ζ.

$$\zeta = \frac{4}{\cancel{t_{s\text{-}2\%}}} \cdot \frac{\cancel{t_{s\text{-}2\%}}\,T_d}{\sqrt{16T_d^2 + (2\pi)^2 t_{s\text{-}2\%}^2}} = \sqrt{\frac{16\,T_d^2}{16T_d^2 + (2\pi)^2 t_{s\text{-}2\%}^2}}$$

$$\zeta = \sqrt{\frac{1}{1 + \left(\frac{2\pi t_{s\text{-}2\%}}{4T_d}\right)^2}} \Longrightarrow \boxed{\zeta = \sqrt{\frac{1}{1 + \left(\frac{\pi t_{s\text{-}2\%}}{2T_d}\right)^2}}}$$

Prob. 4

EQUIVALENT REPRESENTATIONS FOR ζ

$$\boxed{\begin{aligned}
\zeta &= \frac{4}{t_{s\text{-}2\%}\sqrt{\dfrac{16}{t_{s\text{-}2\%}^2} + \left(\dfrac{2\pi}{T_d}\right)^2}} \\
&= \sqrt{\frac{16 T_d^2}{16 T_d^2 + (2\pi)^2 t_{s\text{-}2\%}^2}} \\
&= \sqrt{\frac{1}{1 + \left(\dfrac{2\pi t_{s\text{-}2\%}}{4 T_d}\right)^2}} \\
&= \frac{4 T_d}{\sqrt{16 T_d + \left(2\pi t_{s\text{-}2\%}\right)^2}} \\
&= \frac{2 T_d}{\sqrt{4 T_d + \left(\pi t_{s\text{-}2\%}\right)^2}}
\end{aligned}}$$

Laplace Transform Pairs

f(t)	F(s)	Comment
1. $\delta(t)$	1	Unit impulse
2. 1 for $t>0$	$\frac{1}{s}$	Unit step
3. t for $t>0$	$\frac{1}{s^2}$	Unit ramp
4. e^{-at}	$\frac{1}{s+a}$	Exponential
5. $\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	Sine
6. $\cos \omega t$	$\frac{s}{s^2+\omega^2}$	Cosine
7. $f(t)$	$F(s)$	Function
8. $\frac{df(t)}{dt}$	$sF(s)-f(0^-)$	First Derivative
9. $\frac{d^n f(t)}{dt^n}$	$s^nF(s)-s^{n-1}f(0^-)-s^{n-2}\frac{df(0^-)}{dt}\cdots$	n^{th} Derivative
10. t^n	$\frac{n!}{s^{n+1}}$	
11. $e^{-at}f(t)$	$F(s+a)$	
12. te^{-at}	$\frac{1}{(s+a)^2}$	
13. $t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	
14. $e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	
15. $e^{-at}\cos \omega t$	$\frac{(s+a)}{(s+a)^2+\omega^2}$	
16. Initial Value Theorem:	$f(0^+)=\lim_{s\to\infty} sF(s)$	
17. Final Value Theorem:	$\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$	