ME 375 EXAM #1

Tuesday, February 25, 2014
6:30 – 7:30 pm
EE 129

Division: Chiu(7:30) / Ding(10:30) / Chiu(3:30) (circle one) **HW ID:** ___________

Name: Solution

Instructions

(1) This is a closed book examination, but you are allowed one single-sided 8.5×11 crib sheet.

(2) You have one hour to work all three problems in the exam.

(3) Write your name and HW ID on the top of each page.

(3) Use the solution procedure: what are you given, what are you asked to find, what are your assumptions, what is your solution, does your solution make sense. You must show all of your work to receive any credit.

(4) You must write neatly and should use a logical format to solve the problems. You are encouraged to really "think" about the problems before you start to solve them. Please write your name in the top right-hand corner of each page.

(5) A table of Laplace transform pairs and properties of Laplace transforms will also be handed out.

Problem No. 1 (30%) ______________________

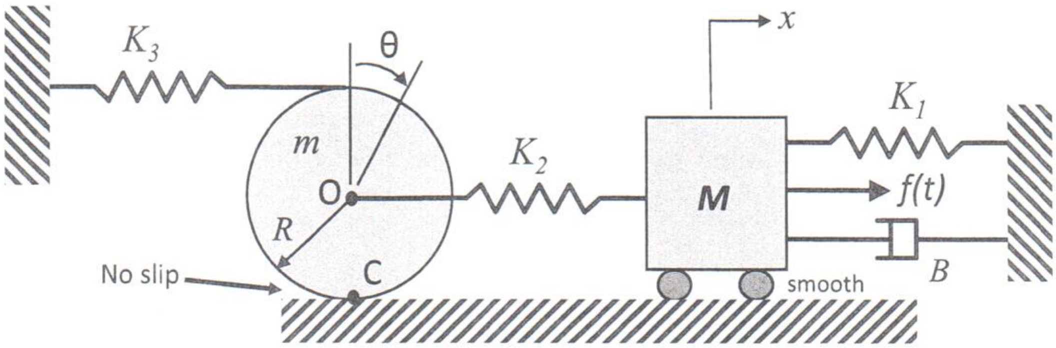
Problem No. 2 (40%) ______________________

Problem No. 3 (30%) ______________________

TOTAL (*/100%)** ______________________

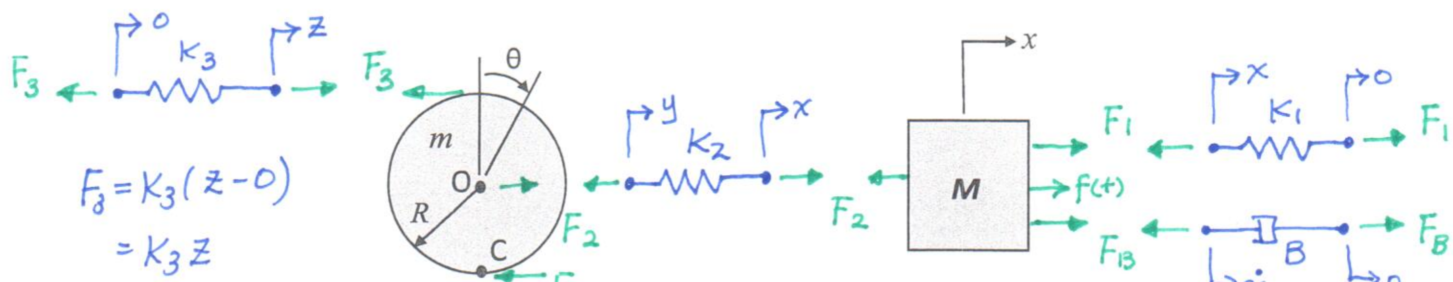
Name: ______________________________ HW ID: ____________

Problem 1 (30%)



The system shown above is made up of a block A (with mass of M) and a homogeneous disk (of mass m and outer radius R) that rolls without slipping on a horizontal surface. A cable is wrapped around the outer radius of the disk with the other end of the cable connected to a spring with spring constant K_3. Three springs and a dashpot connects together the components of the system as shown in the figure. An input force $f(t)$ acts on block A. Let x represent the translation of block A and θ represent the rotation of the disk. The springs are unstretched when $x = \theta = 0$. Assume that the cable remains taut for all motion of the system. The centroidal mass moment of inertia of a homogeneous disk of mass m and outer radius R about axis of rotation through the center O of the disk is given by $I_O = \frac{1}{2}mR^2$

(A) (10%) Draw the free body diagram of the components (on the following figures) and their corresponding elemental equations.



$F_3 = K_3(z-0)$
$= K_3 z$

$F_2 = K_2(x-y)$

$F_1 = K_1(0-x) = -K_1 x$

$F_B = B(0-\dot{x}) = -B\dot{x}$

Moment around instantaneous center of rotation C

$M\ddot{x} = \Sigma F_i = F_1 + F_B + f(t) - F_2$

$I_C\ddot{\theta} = \Sigma T_i$
$= F_2 \cdot R - F_3 \cdot 2R$

Name: ______________________________ HW ID: ____________

Problem 1

(B) (10%) Derive the differential equations of motion (EOMs) for the system in terms of the coordinates x and θ.

$$M\ddot{x} = F_1 + F_B + f(t) - F_2$$

$$I_C\ddot{\theta} = F_2 \cdot R - F_3 \cdot 2R$$

$F_1 = -K_1 x$

$F_B = -B\dot{x}$

$F_2 = K_2(x-y)$

$F_3 = K_3 z$

$I_C = I_O + mR^2$

$= \frac{1}{2}mR^2 + mR^2 = \frac{3}{2}mR^2$

Rolling without slipping:

$$\begin{cases} y = R\theta \\ z = 2R\theta \end{cases}$$

$$\Rightarrow \quad M\ddot{x} = -K_1 x - B\dot{x} + f(t) - K_2(x-y)$$

$$= -(K_1+K_2)x - B\dot{x} + K_2 y + f(t)$$

$$\Rightarrow \quad \boxed{M\ddot{x} + B\dot{x} + (K_1+K_2)x - K_2R\theta = f(t)}$$

$$I_C\ddot{\theta} = F_2 \cdot R - F_3 \cdot 2R = K_2(x-y)\cdot R - K_3 \cdot z \cdot 2R$$

$$= K_2Rx - K_2R^2\theta - K_3 \cdot 2R\theta \cdot 2R$$

$$= K_2Rx - (K_2+4K_3)R^2\theta$$

$$\Rightarrow \quad \boxed{\frac{3}{2}mR^2\ddot{\theta} + (K_2+4K_3)R^2\theta - K_2Rx = 0}$$

OR

$$\boxed{\frac{3}{2}mR\ddot{\theta} + (K_2+4K_3)R\theta - K_2x = 0}$$

Name: ______________________________ HW ID: ____________

Problem 1

(C) (10%) Based on your EOM in (B), assume $K_1 = K_2 = K_3 = K$, determine the single input/output differential equation of motion of the system for output of θ and input $f(t)$. If you did not complete (B) or are not confident about your solution, use the following EOM:

$$M\ddot{x} + 2B\dot{x} + 2Kx - KR\theta = f(t)$$
$$2mR^2\ddot{\theta} + 4KR^2\theta - KRx = 0$$

<u>Solution I</u>: use result from (B)

$$\begin{cases} M\ddot{x} + B\dot{x} + (K_1+K_2)x - K_2R\theta = f(t) \\ \frac{3}{2}mR\ddot{\theta} + (K_2+4K_3)R\theta - K_2x = 0 \end{cases}$$

$K_1 = K_2 = K_3 = K$

$$\Rightarrow \begin{cases} M\ddot{x} + B\dot{x} + 2Kx - KR\theta = f(t) \\ \frac{3}{2}mR\ddot{\theta} + 5KR\theta - Kx = 0 \end{cases} \Rightarrow x = \frac{1}{K}\left(\frac{3}{2}mR\ddot{\theta} + 5KR\theta\right)$$

$$\dot{x} = \frac{1}{K}\left(\frac{3}{2}mR\dddot{\theta} + 5KR\dot{\theta}\right)$$

$$\ddot{x} = \frac{1}{K}\left(\frac{3}{2}mR\ddddot{\theta} + 5KR\ddot{\theta}\right)$$

$$\Rightarrow \frac{1}{K}M\left(\frac{3}{2}mR\ddddot{\theta} + 5KR\ddot{\theta}\right) + \frac{1}{K}B\left(\frac{3}{2}mR\dddot{\theta} + 5KR\dot{\theta}\right) + 2\left(\frac{3}{2}mR\ddot{\theta} + 5KR\theta\right) - KR\theta = f(t)$$

$$\Rightarrow \boxed{\frac{3}{2K}MmR\ddddot{\theta} + \frac{3}{2K}BmR\dddot{\theta} + (5M+3m)R\ddot{\theta} + 5BR\dot{\theta} + 9KR\theta = f(t)}$$

<u>Solution II</u>: use given result

$$\boxed{\frac{2}{K}MmR\ddddot{\theta} + \frac{4}{K}BmR\dddot{\theta} + (4M+4m)R\ddot{\theta} + 8BR\dot{\theta} + 7KR\theta = f(t)}$$

Name: ______________________________ HW ID: ____________

Problem 2 (40%)
A system has following model:

$$\ddot{y} + 4\dot{y} + 5y = 2\dot{u} + 5u\ ,$$

where y is the output and u is the input

(A) (10%) Find the Laplace transform of the free response $Y(s)$, when $y(0) = 1$ and $\dot{y}(0) = 2$. Note that you do not need to fine $y(t)$.

Free response assumes $u(t) = 0$

Take Laplace transform

$$\Rightarrow \quad s^2 Y(s) - s\,y(0) - \dot{y}(0) + 4[s Y(s) - y(0)] + 5 Y(s) = 0$$

$$\Rightarrow \quad (s^2 + 4s + 5) Y(s) = (s+4)\,y(0) + \dot{y}(0)$$
$$= (s+4)\cdot 1 + 2$$
$$= (s+6)$$

$$\Rightarrow \quad \boxed{Y_{Free}(s) = \frac{s+6}{(s^2+4s+5)}}$$

(B) (10%) Write the transfer function between input u and output y for the above model.

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{2s+5}{s^2+4s+5}}$$

Name: ______________________ HW ID: __________

Problem 2

(C) (10%) Find the forced response of $y(t)$ for unit step input and $u(0) = 0$

Forced response, assumes zero I.C.'s

$$Y_{Forced}(s) = G(s) \cdot U(s) = \frac{2s+5}{s^2+4s+5} \cdot \frac{1}{s}$$

$$= \frac{2s+5}{s\left((s+2)^2+1^2\right)}$$

$$= \frac{A}{s} + \frac{B(s+2)}{(s+2)^2+1^2} + \frac{C \cdot 1}{(s+2)^2+1^2}$$

$$\Rightarrow A(s^2+4s+5) + B(s+2) \cdot s + C \cdot s = 2s+5$$

$$\Rightarrow \begin{cases} A+B=0 \\ 4A+2B+C=2 \\ 5A=5 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \\ C=0 \end{cases}$$

$$\Rightarrow Y_{Forced}(s) = \frac{1}{s} - \frac{s+2}{(s+2)^2+1^2}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left[Y_{Forced}(s)\right]$$

$$= 1 - e^{-2t}\cos(t)$$

$$\Rightarrow \boxed{y(t) = 1 - e^{-2t}\cos(t)}$$

Name: ______________________________ HW ID: ____________

Problem 2

(D) (10%) Write down the state space form for $\ddot{y}+4\dot{y}+5y=5u$

Define state variables

$$\begin{cases} x_1 = y \\ x_2 = \dot{y} \end{cases}$$

state equation:

$$\begin{cases} \dot{x}_1 = \dot{y} = x_2 \\ \dot{x}_2 = \ddot{y} = -5y - 4\dot{y} + 5u \\ \qquad = -5x_1 - 4x_2 + 5u \end{cases}$$

output equation

$$y = x_1$$

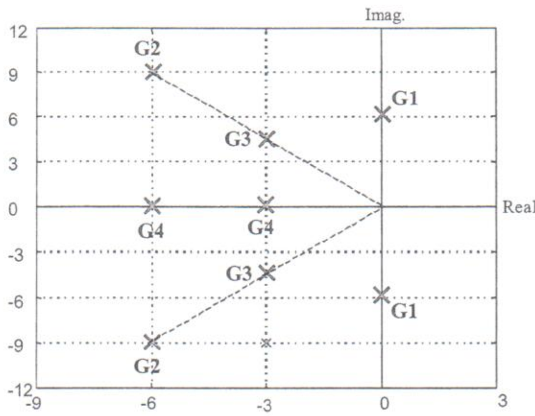
State space ~~ne~~ form:

$$\boxed{\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}}$$

Name: ______________________________ HW ID: ____________

Problem 3 (30%)

(A) (20%) The pole positions of four second order systems are shown below. The unit step responses of these systems are also shown. Match up the pole positions (labeled *G*1, *G*2, *G*3, *G*4) with the correct unit step response (labeled *R*1, *R*2, *R*3, *R*4). (Hint: consider settling time and overshoot)

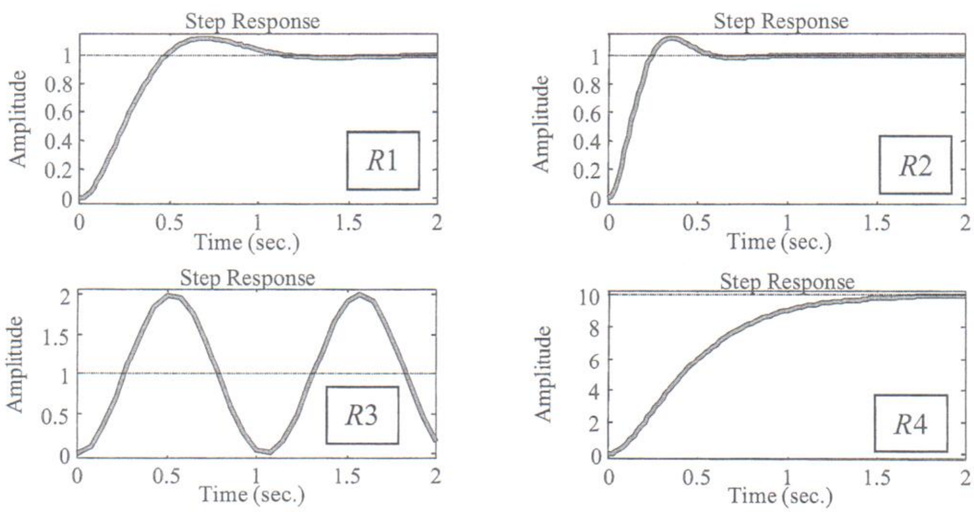


*G*1 ⇔ R3

*G*2 ⇔ R2

*G*3 ⇔ R1

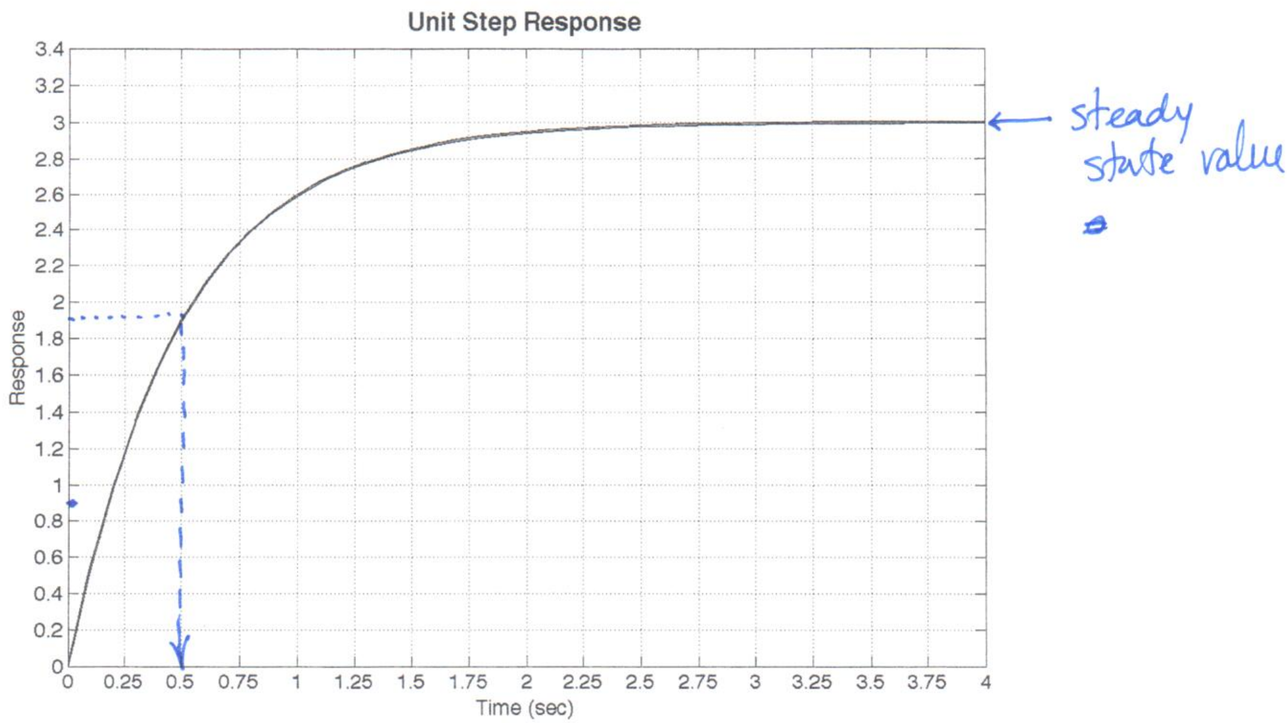
*G*4 ⇔ R4



Name: ______________________________ HW ID: ______________

Problem 3

(B) (10%) The unit step response of a first order system $\tau\dot{y} + y = K \cdot u$ is given below. Use this plot to estimate the time constant τ of the first order system and the static gain K of the system. Mark up on the figure where you use the values/information to make your estimation.



$y_{ss} = 3$ $\qquad$ $y_{ss} = K \cdot u = K$

$\Rightarrow \boxed{K = 3}$

63.2% of $y_{ss} \Rightarrow 0.632 \cdot 3 = 1.896 \approx 1.9$

$\boxed{\tau \cong 0.5}$