ME 375 EXAM #2
Tuesday, April 5, 2005

Division King 11:30 / Cunningham 2:30 (circle one) **Name** SOLUTION

Instructions

(1) This is a closed book examination, but you are allowed two 8.5×11 crib sheets.

(2) You have one hour to work all three problems on the exam.

(3) Use the solution procedure we have discussed: what are you given, what are you asked to find, what are your assumptions, what is your solution, does your solution make sense. You must show all of your work to receive <u>any</u> credit.

(4) Circle or box-in your answers.

(5) You must write neatly and should use a logical format to solve the problems. You are encouraged to really "think" about the problems before you start to solve them. Please write your name in the top right-hand corner of each page.

(6) A table of Laplace transform pairs and properties of Laplace transforms is attached at the end of this exam set.

Problem No. 1 (30%)	30
Problem No. 2 (30%)	30
Problem No. 3 (40%)	40
TOTAL (*/100%)**	100

PROBLEM NO. 1 (30%)

Consider the second-order transfer function given below. Assume $y(0)=0$ and $\dot{y}(0)=0$.

$$\frac{Y(s)}{F(s)} = G(s) = \frac{K_s \cdot \omega_n^2}{s^2 + 2\zeta\omega_n \cdot s + \omega_n^2}$$

(a) Given the input $f(t) = Au(t)$, where $u(t)$ is the unit step function, solve for the dynamic response, $y(t)$, and clearly identify the transient and steady-state portions of the response.

$$Y(s) = \left(\frac{K_s \omega_n^2}{s^2 + 2\zeta\omega_n \cdot s + \omega_n^2}\right)\left(\frac{A}{s}\right) = \frac{A_1}{s} + \frac{A_2(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2 + \omega_d^2} + \frac{A_3\,\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

$$= \frac{A_1(s^2 + 2\zeta\omega_n s + \omega_n^2) + A_2 s(s+\zeta\omega_n) + A_3 s(\omega_d)}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$s^2:\ A_1 + A_2 = 0 \qquad \Rightarrow A_2 = -A_1 = -K_s A$$

$$s^1:\ (2\zeta\omega_n)A_1 + (\zeta\omega_n)A_2 + (\omega_d)A_3 = 0 \Rightarrow A_3 = \frac{K_s}{\omega_d}(\zeta\omega_n - 2\zeta\omega_n) = -\frac{\zeta\omega_n}{\omega_d}K_s A$$

$$s^0:\ (\cancel{\omega_n^2})A_1 = K_s \cancel{\omega_n^2} A \qquad \Rightarrow A_1 = K_s \cdot A$$

$\mathcal{L}^{-1}(\cdot)$

$$\boxed{y(t) = \Big[\underbrace{K_s \cdot A}_{\text{steady state}} - \underbrace{K_s A e^{-\zeta\omega_n t}\cos(\omega_d t) - \frac{\zeta\omega_n}{\omega_d} K_s A e^{-\zeta\omega_n t}\sin(\omega_d t)}_{\text{transient}}\Big]u(t)}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$

(b) Using the result from part (a), find the numerical value of the product $(\zeta\omega_n)$ to ensure that $\frac{y_{ss} - y(t)}{y_{ss}} \le 0.05$ for $t \ge 1$ second.

$$y_{ss} = K_s A$$

$$y(1) = K_s A - K_s A e^{-\zeta\omega_n}\overbrace{\left[\cos(\omega_d) + \frac{\zeta\omega_n}{\omega_d}\sin(\omega_d)\right]}^{0 \le |\cdot| \le 1}$$

$$\frac{y_{ss} - y(1)}{y_{ss}} = \frac{\cancel{K_s}A - \cancel{K_s}A + \cancel{K_s}A e^{-\zeta\omega_n}\left[\cos(\omega_d) + \frac{\zeta\omega_n}{\omega_d}\sin(\omega_d)\right]}{\cancel{K_s}A}$$

$$= e^{-\zeta\omega_n}[1] \le 0.05$$

$$\ln(e^{-\zeta\omega_n}) \le \ln(0.05) \Rightarrow \boxed{(\zeta\omega_n) \ge \ln\left(\frac{1}{0.05}\right)} = 3$$

OR 5% SETTLING TIME

$$3\tau = \frac{3}{\zeta\omega_n} \le 1\text{ sec}$$

$$\boxed{(\zeta\omega_n) \ge 3}$$

(c) Consider a different input to $G(s)$,

$$f(t) = 3.5 + 11.71\cos(4.26t) - 6.87\sin\left(15.33t + \frac{\pi}{3}\right).$$

When $K_s = 7.56$ $\zeta = 0.479$, and $\omega_n = 4.26$, find the steady-state response, y_{ss}, to $f(t)$.

$$G(j\omega) = \frac{K_s \omega_n^2}{-\omega^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \frac{K_s}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right] + j\left(2\zeta\frac{\omega}{\omega_n}\right)}$$

$$|G(j\omega)| = \frac{K_s}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

$$\measuredangle G(j\omega) = 0 - \operatorname{atan}\left(\frac{2\zeta\,\omega/\omega_n}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right) = -\operatorname{atan}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)$$

ω	$\lvert G(j\omega)\rvert$	$\measuredangle G(j\omega)$
0	$K_s = 7.56$	0
$\omega_n = 4.26$	$K_s/2\zeta = 7.89$	$-90° = -\frac{\pi}{2}$ rad
15.33	0.608	$-164° = -2.86$ rad

$$y_{ss}(t) = 3.5\,|G(j0)| + (11.71)\,|G(j4.26)|\cos\left(4.26t + \measuredangle G(j4.26)\right)$$
$$- (6.87)\,|G(j15.33)|\sin\left(15.33t + \frac{\pi}{3} + \measuredangle G(j15.33)\right)$$

$$\boxed{\begin{aligned} y_{ss}(t) &= 26.46 + 92.41\cos\left(4.26 - \frac{\pi}{2}\right) \\ &\quad - 4.18\sin\left(15.33 + \frac{\pi}{3} - 2.86\right)\end{aligned}}$$

PROBLEM NO. 2 (30%)

Draw the asymptotic bode plot approximation for the transfer function,

$$G(s) = \frac{0.0255 \cdot s + 0.0382}{s^3 + 0.0380 \cdot s^2 + 0.0036 \cdot s}, \quad = \frac{\overbrace{0.0382}^{(\overbrace{10.61}^{K_s})(\overbrace{0.0036}^{\omega_n^2})}\,(0.667s+1)}{s\,(s^2 + \underbrace{0.0380}_{2\zeta\omega_n}\,s + \underbrace{0.0036}_{\omega_n^2})}$$

using the following steps:

(a) Find the poles, zeros, and the static gain of the transfer function $G(s)$.

zero: -1.5

poles: $-0.019 \pm j\,0.0569$, 0

static gain: $G(0) \to \infty$ (b/c of free integrator)

(b) Extract the elemental transfer functions writing all first-order terms in the form $(\tau \cdot s + 1)$ (except for s or $1/s$) and all the second-order terms in the form $\omega_n^2 / (s^2 + 2\zeta\omega_n \cdot s + \omega_n^2)$.

$G_0 = K_s = 10.61 \Rightarrow 20.5\text{dB} = 20\log_{10}(10.61)$

$G_1 = \left(\frac{s}{1.5} + 1\right)$

$G_2 = \frac{1}{s}$

$G_3 = \dfrac{0.0036}{s^2 + 0.0380\,s + 0.0036}$

(c) Identify the break frequencies of each of the elemental transfer functions comprising $G(s)$.

$\omega_{b0} \to \infty$ $\qquad \omega_{b2} \to 0$ $\qquad$ optional

$\boxed{\omega_1 = 1.5 \qquad \omega_3 = 0.06}$ required

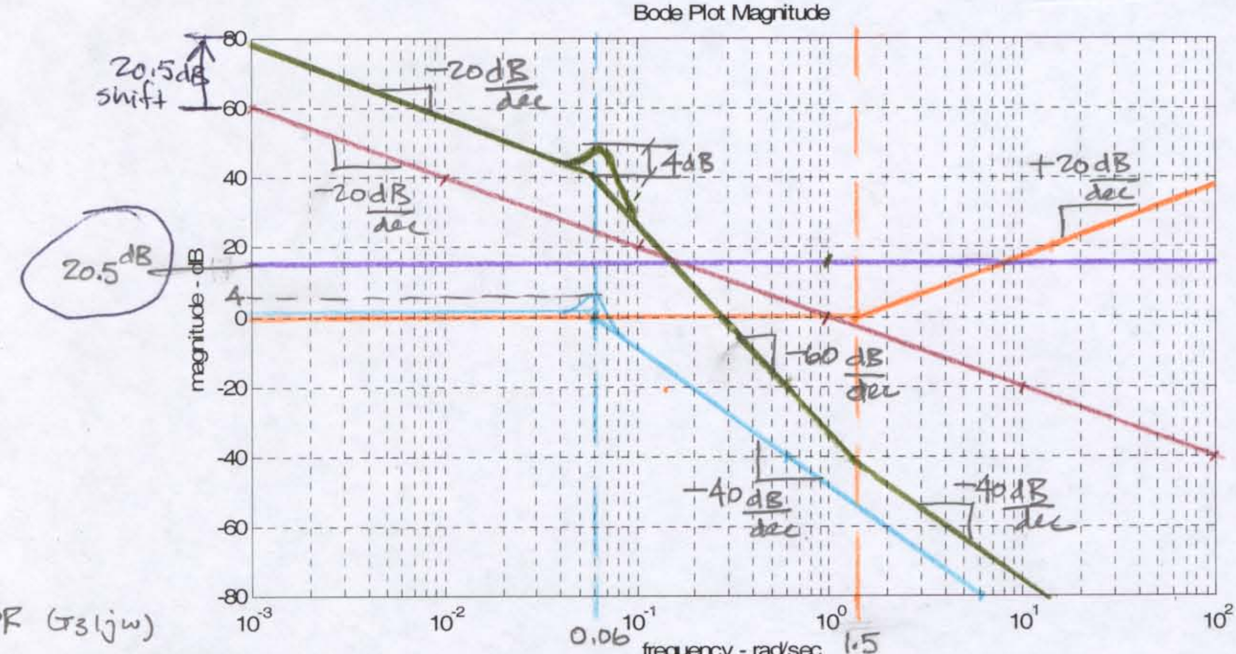
(d) Plot the asymptotic magnitude and phase plots for each of the elemental transfer functions on the graphs on the next page. Be sure to clearly label each of the elemental transfer functions on the graph along with all break frequencies and slopes.

(e) Plot the total transfer function, $G(s)$, from the composite of the elemental transfer functions in part (d).

$G_0(j\omega)$ — $G_2(j\omega)$

$G_1(j\omega)$ — $G_3(j\omega)$ — $G(j\omega)$ TOTAL BODE DIAGRAM

Bode Plot Magnitude

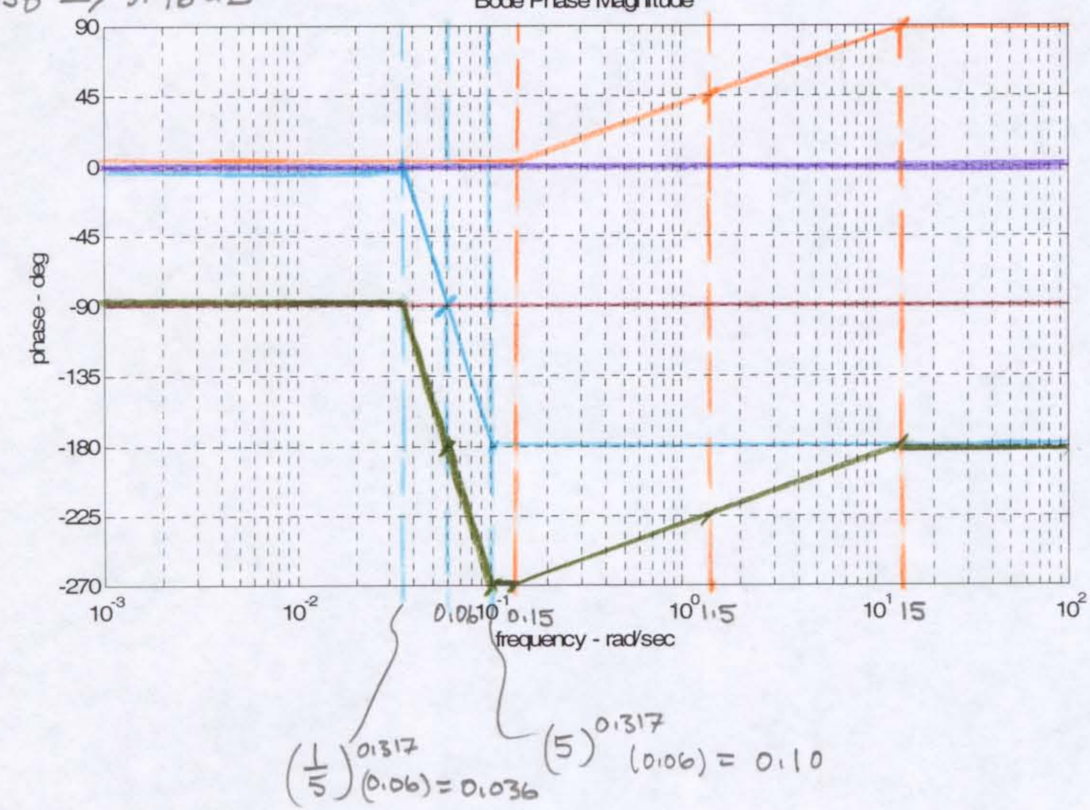


FOR $G_3(j\omega)$

$$\left.\begin{array}{l} 2\zeta\omega_n = 0.0380 \\ \omega_n = 0.06 \end{array}\right\} \Rightarrow \zeta = 0.317$$

$$M_r \approx \frac{1}{2\zeta} = 1.58 \Rightarrow 3.96\text{ dB}$$

Bode Phase Magnitude



PROBLEM No. 3: (40%)

In lecture the following Input / Output Model for a DC electric motor was developed.

$E_i(s) = E_A(s)$

$$\Omega(s)=\frac{K_T E_i(s)}{L_A J_A s^2+(L_A B+R_A J_A)s+(R_A B+K_b K_T)}-\frac{(L_A s+R_A)T_L}{L_A J_A s^2+(L_A B+R_A J_A)s+(R_A B+K_b K_T)}$$

where J_A is the armature moment of inertia, K_T is the torque constant, K_b is the back emf constant, R_A is armature resistance, L_A is the armature inductance, T_L is the load torque, B is the armature friction, and Ω is the armature speed.

(a) Assuming that B is small develop the following:

(i) $\frac{K_T}{R_A}=\frac{T_{stall}}{e_a}$

FOR T_{stall} (STALL TORQUE) $\Omega = 0$

AND THIS IS STEADY STATE

$$\Omega = 0 = \frac{K_T E_A}{\cancel{L_A J_A s^2}^{\,0} + (L_A B + \cancel{J_A R_A}^{\,0})s + (R_A B + K_b K_T)}$$

AT S.S. $s=0 \Rightarrow$

$$- \frac{(\cancel{L_A s}^{\,0} + R_A)T_{L\text{-}stall}}{\cancel{L_A J_A s^2}^{\,0} + (L_A B + \cancel{R_A J_A}^{\,0})s + (R_A B + K_b K_T)}$$

$$\frac{K_T E_A}{\cancel{R_A B + K_b K_T}} = \frac{R_A T_{L\text{-}STALL}}{\cancel{R_A B + K_b K_T}} \Rightarrow \boxed{\frac{K_T}{R_A} = \frac{T_{stall}}{E_A}}$$

(ii) $K_b=\frac{e_a}{\omega_{no-load}}$

FOR $\omega_{no\text{-}load}$ (NO-LOAD SPEED) $T_L = 0$

AND STEADY-STATE ($s=0$)

$$\omega_{no\text{-}load} = \frac{K_T E_A}{\cancel{J_A L_A s^2}^{\,0} + (L_A B + \cancel{R_A J_A}^{\,0})s + R_A B + K_b K_T}$$

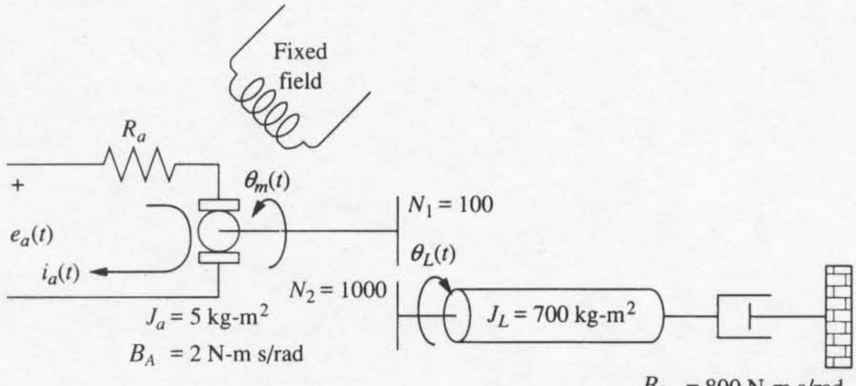
$$\omega_{no\text{-}load} = \frac{\cancel{K_T} E_A}{\cancel{R_A B}^{\,0} + K_b \cancel{K_T}} = \frac{E_A}{K_b} \Rightarrow \boxed{K_b = \frac{E_A}{\omega_{no\text{-}load}}}$$

B IS small

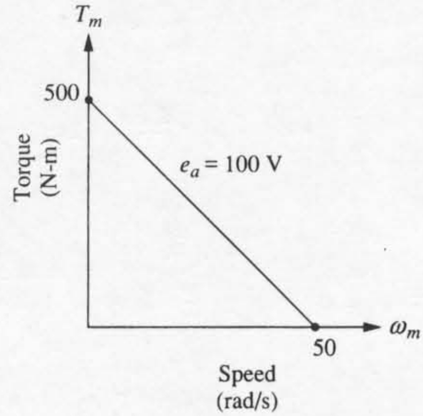
If the transfer function for an electric motor loaded through a transmission is given by the following:

$$\frac{\Omega_m(s)}{E_a(s)}=\frac{K_T/R_A J_{eq}}{(s+\frac{1}{J_{eq}}(B_{eq}+\frac{K_T K_b}{R_A}))}$$ where Jeq and Beq are the effective moment of inertia and

friction for the motor, transmission, and load system. Find the following from the given motor speed map data:



(*a*)



(*b*)

Hint: $J_{eq} = J_A + J_L\left(\frac{N_1}{N_2}\right)^2$ and $B_{eq} = B_A + B_L\left(\frac{N_1}{N_2}\right)^2$

(b) Find the transfer function $\Omega_m(s)/E_a(s)$, where Ω_m is the motor speed. (Put values for the terms into you TF for credit .)

$$J_{eq} = J_A + J_L\left(\frac{N_1}{N_2}\right)^2 = 5 + 700\left(\frac{1}{10}\right)^2 = 12$$

$$B_{eq} = B_A + B_L\left(\frac{N_1}{N_2}\right)^2 = 2 + 800\left(\frac{1}{10}\right)^2 = 10$$

FROM THE TORQUE SPEED MAP: $T_{STALL} = 500$, $\omega_{no\text{-}load} = 50$, $e_a = 100$

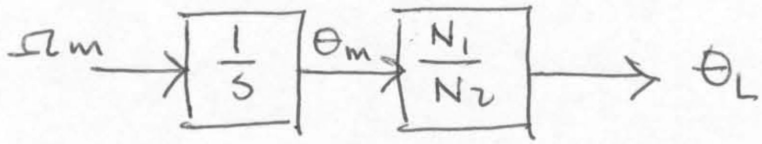
$$\frac{K_T}{R_A} = \frac{T_{stall}}{e_A} = \frac{500}{100} = 5 \quad , \quad K_b = \frac{e_a}{\omega_{no\text{-}load}} = \frac{100}{50} = 2$$

$$\boxed{\frac{\Omega_m(s)}{E_a(s)} = \frac{\frac{5}{12}}{\left[s + \frac{1}{12}\left(10 + 5 \cdot 2\right)\right]} = \frac{0.417}{s + 1.667}}$$

(c) Find the transfer function $\theta_L(s)/E_a(s)$, where θ_L is the load position. (Put values for the terms into you TF for credit .)

$$N_2 \theta_L = N_1 \theta_m$$

$$\frac{\Omega_m}{s} = \theta_m$$



$$\frac{\theta_L(s)}{E_a(s)} = \left(\frac{N_1}{N_2} \cdot \frac{1}{s}\right) \frac{\Omega_m(s)}{E_a(s)} = \frac{1}{10} \cdot \frac{1}{s} \left(\frac{0.417}{s+1.667}\right)$$

$$\boxed{\frac{\theta_L(s)}{E_a(s)} = \frac{0.0417}{s(s+1.667)}}$$

Laplace Transform Pairs

	f(t)	F(s)	Comment
1.	$\delta(t)$	1	Unit impulse
2.	1 for t>0	$\frac{1}{s}$	Unit step
3.	t for t>0	$\frac{1}{s^2}$	Unit ramp
4.	e^{-at}	$\frac{1}{s+a}$	Exponential
5.	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	Sine
6.	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	Cosine
7.	$f(t)$	$F(s)$	Function
8.	$\frac{df(t)}{dt}$	$sF(s)-f(0^-)$	First Derivative
9.	$\frac{d^nf(t)}{dt^n}$	$s^nF(s)-s^{n-1}f(0^-)-s^{n-2}\frac{df(0^-)}{dt}\cdots$	n^{th} Derivative
10.	t^n	$\frac{n!}{s^{n+1}}$	
11.	$e^{-at}f(t)$	$F(s+a)$	
12.	te^{-at}	$\frac{1}{(s+a)^2}$	
13.	t^ne^{-at}	$\frac{n!}{(s+a)^{n+1}}$	
14.	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	
15.	$e^{-at}\cos \omega t$	$\frac{(s+a)}{(s+a)^2+\omega^2}$	
16.	Initial Value Theorem:	$f(0^+)=\lim_{s\to\infty} sF(s)$	
17.	Final Value Theorem:	$\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$	