**PROBLEM 1**: (40%)

Consider a 2nd-order dynamic system $G(s) = \frac{1}{s^2 + 3s + K + 1}$, where $K > 0$ is an unknown parameter.

(a) Find the expressions of damping ratio $\varsigma$, natural frequency $\omega_n$ and static gain $K_s$ in terms of the unknown parameter $K$.

$$K_s = \frac{1}{K+1}$$

$$\omega_n = \sqrt{K+1}$$

$$\varsigma = \frac{3}{2\sqrt{K+1}} \qquad (2\varsigma\omega_n = 3)$$

(b) Determine the range of $K$ such that the system has two complex conjugate poles which satisfy the following two performance requirements: (i) *OS* of the unit step response is no more than *30%*; (ii) *2%* settling time of the unit step response is no greater than *4sec.*

$$s_{1,2} = \frac{-3 \pm \sqrt{5-4K}}{2}$$

two complex poles: $5 - 4K < 0 \Rightarrow K > \frac{5}{4} = 1.25$

$$s_{1,2} = \frac{-3 \pm j\sqrt{4K-5}}{2}$$

$t_s|_{2\%} = \frac{4}{\varsigma\omega_n} = \frac{8}{3} < 4$ sec. $\therefore$ always satisfied.

$OS \leq 30\% \Rightarrow \varsigma \geq \frac{|\ln 0.3|}{\sqrt{\pi^2 + (\ln 0.3)^2}} = 0.36$

i.e. $\frac{3}{2\sqrt{K+1}} \geq 0.36 \Rightarrow K \leq 16.36$

$\therefore \quad 1.25 < K \leq 16.36$

**PROBLEM 1 (Continued)**

(c) When $K = 1$, find out the time-domain expression of the steady-state response of the system to an input signal $u(t) = 2 + 3\sin(5t + \pi/4)$.

$$y(t) = 2\,G(0) + 3\,|G(5j)|\,\sin\left(5t + \frac{\pi}{4} + \angle G(5j)\right)$$

where $G(0) = \frac{1}{2}$ $\quad G(5j) = \frac{1}{(1+5j)(2+5j)}$

$$|G(5j)| = \frac{1}{\sqrt{26}\cdot\sqrt{29}} = 0.036$$

$$\angle G(5j) = 0 - \tan^{-1}5 - \tan^{-1}\frac{5}{2} = -2.56 \text{ rad.}$$

$$\therefore\ y(t) = 1 + 0.1\sin(5t - 1.78)$$

(d) When $K = 8$, find out the resonant (peak) frequency $\omega_r$ of the system and the corresponding peak magnitude $|G(j\omega_r)|$.

$$\omega_r = \omega_n\sqrt{1-2\zeta^2} = 3\cdot\sqrt{1-2\cdot 0.5^2} = 2.12 \text{ rad/s}$$

$$|G(j\omega_r)| = \frac{|G(0)|}{2\zeta\sqrt{1-\zeta^2}} = \frac{\frac{1}{9}}{2\times 0.5\sqrt{1-0.5^2}} = 0.13$$

**PROBLEM 2** (30%)

Given the transfer function $G(s) = \dfrac{8(s+20)s}{\left(s^2+2s+16\right)}$, draw the Bode straight line approximation using the following steps:

(a) Write all first order terms in the form ($\tau s + 1$) (except $s$ or $1/s$) and all second order complex terms in the form $\dfrac{\omega_n{}^2}{s^2+2\zeta\omega_n s+\omega_n{}^2}$.

$$G(s) = \underbrace{\frac{16}{s^2+2s+16}}_{G_1} \cdot \underbrace{\left(\frac{s}{20}+1\right)}_{G_2} \cdot \underbrace{s}_{G_3} \cdot \underbrace{10}_{G_4}$$

$$G_1(s) = \frac{16}{s^2+2s+16}$$

$$G_2(s) = \left(\frac{s}{20}+1\right)$$

(b) Provide the break frequencies for the first and second order terms in transfer function $G(s)$.

$$\omega_1 = 4$$

$$\omega_2 = 20$$

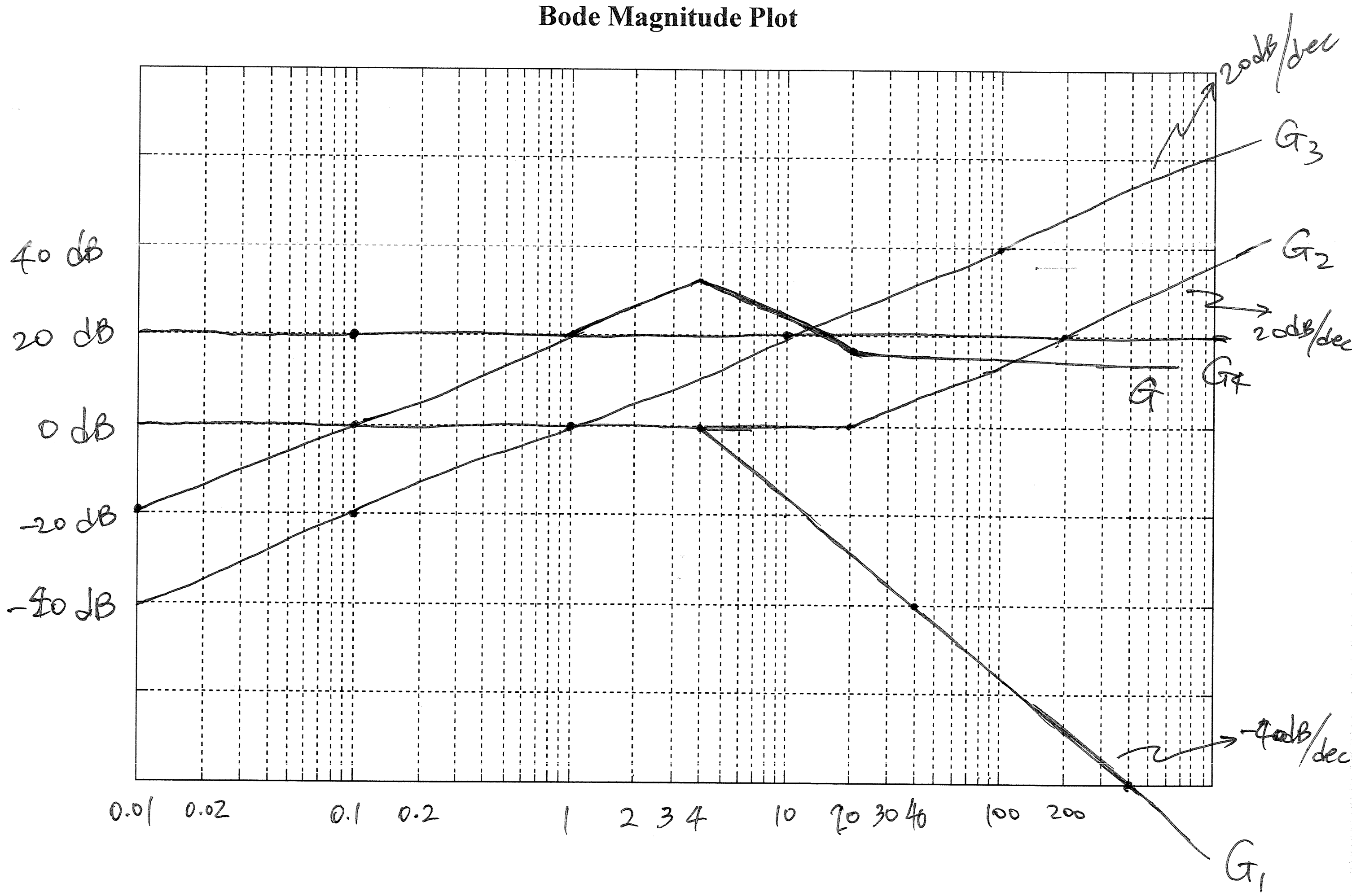
**PROBLEM 2 (Continued)**

(c) Plot the straight line approximation for the magnitude of all elemental transfer functions separately on the graphs on this page. You do NOT need to include resonant peaks in the magnitude plot for any second order terms.

Label the elemental transfer functions on the graph along with all break frequencies and slopes.

(d) Plot the Bode magnitude plot of the total transfer function made up of the elemental transfer functions in part (c).

**Bode Magnitude Plot**



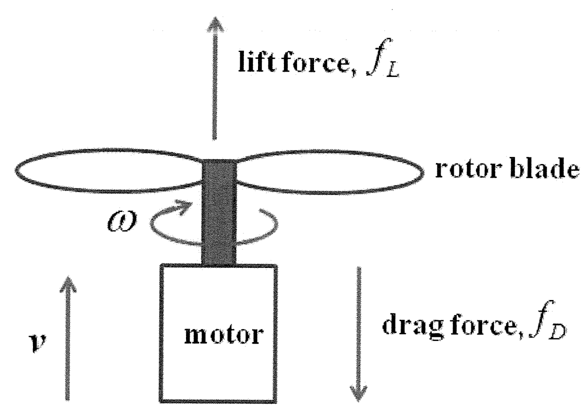
**PROBLEM 3**: (30%)

The coupled electro-mechanical equations for an electric motor (ignoring armature inductance and armature damping) are given by:

$$J\dot{\omega} = K_T i_a - \tau_L$$
$$e_i = R i_a + K_b \omega$$

where $\omega$ is the rotation rate of the motor, $J$ is the mass moment of inertia of the armature, $K_T$ is the motor torque constant, $K_b$ is the back EMF constant, $i_a$ is the armature current, $\tau_L$ is the load torque, $e_i$ is the input voltage applied to the armature and $R$ is the armature resistance.

An electric motor governed by the above equations is used to propel a pair of rotor blades to lift the weight of the vehicle consisting of the motor and rotor blades. For example, it can be one of the four rotors of a quadrotor helicopter. In this application, the rotor blades exert a resistive load torque of $\tau_L = K_{Load}\omega$ on the output shaft of the motor, and the rotation of the rotor blades creates a lift force $f_L = K_{Lift}\omega$ to sustain the weight of the vehicle (where $K_{Load}$ and $K_{Lift}$ are constants). In addition, the vehicle experiences a drag force of $f_D = K_D v$ where $v$ is the speed of the vehicle and $K_D$ is a constant. Let $m$ represent the total mass of the vehicle consisting of the motor and rotor blades.

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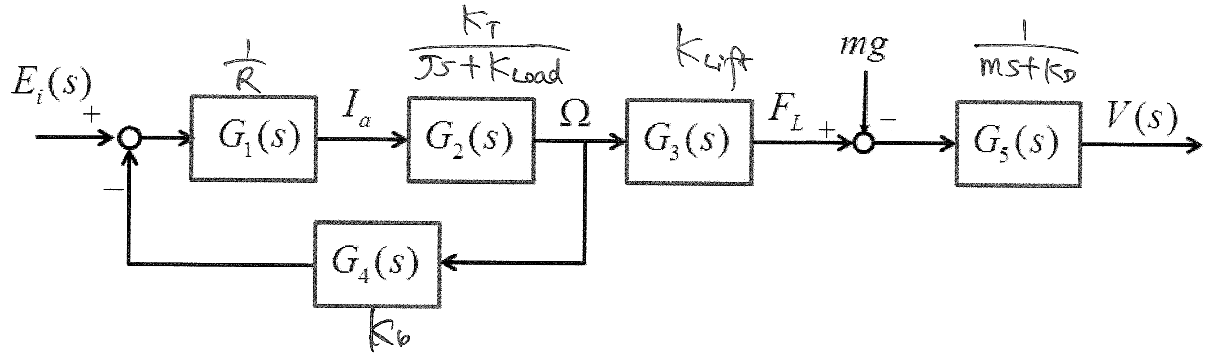
**PROBLEM 3**: (continued)

For this problem:

(a) The general form of the block diagram for the vehicle system is shown below. DERIVE the transfer functions $G_1(s)$, $G_2(s)$, $G_3(s)$, $G_4(s)$, and $G_5(s)$ shown in this block diagram.




$$J\dot{\omega} = K_T i_a - \tau_L = K_T i_a - K_{Load}\,\omega$$

$$\mathcal{L}: \quad Js\Omega(s) = K_T I_a(s) - K_{Load}\,\Omega(s)$$

$$\therefore \quad \Omega(s) = \underbrace{\frac{K_T}{Js + K_{Load}}}_{G_2(s)} I_a(s)$$

$$V(s) = \underbrace{\frac{1}{ms + K_D}}_{G_5(s)}\left[K_{Lift}\,\Omega - mg(s)\right]$$

$$e_i = R i_a + K_b \omega$$

$$\mathcal{L}: \quad E_i(s) = R I_a(s) + K_b \Omega(s)$$

$$\therefore \quad I_a(s) = \underbrace{\frac{1}{R}}_{G_1(s)}\left[E_i(s) - \underbrace{K_b}_{G_4(s)}\Omega(s)\right]$$

$$m\dot{v} = \Sigma F = f_L - f_D - mg = K_{Lift}\,\omega - K_D v - mg$$

$$\mathcal{L}: \quad msV(s) = \underbrace{K_{Lift}}_{G_3(s)}\Omega(s) - K_D V(s) - mg(s)$$

**PROBLEM 3**: (continued)

(b) Develop the transfer function relating the input voltage $e_i$ to the vehicle speed $v$, $G(s) = \dfrac{V(s)}{E_i(s)}$

$$G(s) = \frac{G_1 G_2 G_3 G_5}{1 + G_1 G_2 G_4}$$

$$= \frac{\frac{1}{R} \cdot \frac{K_T}{Js + K_{Load}} \cdot K_{Lift} \cdot \frac{1}{ms + K_D}}{1 + \frac{1}{R} \cdot \frac{K_T}{Js + K_{Load}} \cdot K_b}$$

$$= \frac{K_T K_{Lift}}{(JRs + K_{Load} R + K_T K_b)(ms + K_D)}$$

(c) When the vehicle is hovering at a constant height in air, what is the input voltage?

$v = 0$ when hovering. (or $f_{Lift} = mg$)

$$V(s) = G_{VE_i}(s) \cdot E_i(s) - \frac{1}{ms + K_D} \cdot \frac{mg}{s}$$

$V_{ss} = \lim_{s \to 0} s \cdot V(s) = 0$ at hovering, when $E_i(s) = \dfrac{e_i}{s}$;

$$\lim_{s \to 0} sV(s) = \frac{K_T K_{Lift} \cdot e_i}{(K_{Load} R + K_T K_b) \cdot K_D} - \frac{mg}{K_D} = 0$$

$$\Rightarrow e_i = \frac{(K_{Load} R + K_T K_D)\, mg}{K_T K_{Lift}}$$