ME 375 FINAL EXAM SOLUTIONS
Friday December 17, 2004

Division Adams 10:30 / Yao 12:30 (circle one)  Name_____________________

Instructions

(1) This is a closed book examination, but you are allowed three 8.5x11 crib sheet.

(2) You have two hours to work all six problems on the exam.

(3) You must show all of your work to receive any credit.

(4) You must write neatly and should use a logical format to solve the problems. You are encouraged to really “think” about the problems before you start to solve them. Please write your name in the top right-hand corner of each page.

Problem No. 1 (10%) ________________________

Problem No. 2 (15%) ________________________

Problem No. 3 (20%) ________________________

Problem No. 4 (30%) ________________________

Problem No. 5 (15%) ________________________

Problem No. 6 (10%) ________________________

TOTAL (***/100%) ________________________
PROBLEM 1 (10%):

Consider the feedback system below.

\[
\begin{align*}
\text{R}(s) & \rightarrow 10 \rightarrow \frac{10}{s(s+a)} \rightarrow \text{Y}(s)
\end{align*}
\]

(a) Show that the closed loop characteristic equation is given by:

\[
1 + a \frac{s}{s^2 + 100} = 0
\]

The closed loop transfer function between \( R(s) \) and \( Y(s) \) is given by:

\[
\frac{Y(s)}{R(s)} = \frac{100}{s^2 + as + 100}
\]

Therefore the characteristic equation is:

\[
s^2 + as + 100 = 0
\]

which can be rearranged to the following:

\[
1 + a \frac{s}{s^2 + 100} = 0
\]
(b) Draw the root locus for this closed loop system as the parameter, $a$, varies from 0 to infinity and calculate and provide all of the root locus information required to do this (open loop poles, open loop zeros, break-in points, asymptotes, etc.).

From the characteristic equation, the open loop poles are at $\pm j10$ rad/s and the open loop zero is at $s=0$ rad/s. Therefore, there are two root loci and they start at the two poles and end at the zero and infinity.

In order to end at the origin and infinity, the two root loci must break-in to the real axis at the following location:

\[
\begin{align*}
    a \text{ such that } \frac{da}{ds} &= \frac{d}{ds} \left( -\frac{s^2 + 100}{s} \right) = 0 \\
    0 &= -\frac{2s \cdot s - 1 \cdot (s^2 + 100)}{s^2} \\
    0 &= -\frac{2s^2 - s^2 - 100}{s^2} \\
    0 &= -s^2 + 100 \\
    s_{1,2} &= \pm 10
\end{align*}
\]

The root at -10 is the break-in point.

At this state, the only other root locus rule needed is that there must be root locus on the real axis to the left of an odd number of real poles/zeros; therefore, loci must be drawn on the real axis to the left of the origin. The final result is shown below.
PROBLEM 2 (15%)

Consider the control system shown below where $Y_d(s)$ is the Laplace transform of the desired output, $E(s)$ is the control error, $Y(s)$ is the Laplace transform of the actual output, $U(s)$ is the Laplace transform of the control input, $D(s)$ is the disturbance, and $G_c(s)$ is the compensator. Answer the following series of questions.

(a) You are asked to stabilize the steady state response of the closed loop system to a step input. First, use a proportional control law, $G_c(s) = K_p$, and plot the root locus as $K_p$ varies from 0 to $\infty$ on the grid provided below. Then comment on whether or not you can stabilize the system using proportional control and, if so, give the stabilizing range of $K_p$ (5 pts).

We plot the roots of the closed loop characteristic equation, $1 + K_p \frac{s}{s - 1} = 0$, by first finding the open loop poles and zeros and then drawing root locus to the left of an odd number of real poles and/or zeros.

We cannot stabilize the system using proportional control because the roots never pass into the left half plane. The best we can do is to place the pole at the origin, which will provide an infinite steady state response to a step input.
(b) Assume a control law having the form of \( G(s) = K/(s-0.5) \). Plot the root locus as \( K \) varies from 0 to \( \infty \) on the grid provided below. Then comment on whether or not you can stabilize the system using such a control law. If not, why. If yes, what is the range of \( K \) that stabilizes the closed-loop system.

We plot the roots of the closed loop characteristic equation, \( 1 + \frac{K}{s-0.5} \frac{s}{s-1} = 0 \), by first finding the open loop poles and zeros and then drawing root locus to the left of an odd number of real poles and/or zeros. Then we find the break away and break in points as follows:

\[
K = -\frac{(s-1)(s-0.5)}{s} = -\frac{s^2 - 1.5s + 0.5}{s}
\]

\[
dK \frac{ds}{ds} = -\left(2s - 1.5\right)s - \left(s^2 - 1.5s + 0.5\right) = 0
\]

\[
s_{1,2} = \pm\sqrt{0.5} = \pm0.707
\]

The imaginary axis crossing can be found by substituting \( s = j\omega \) into the closed-loop characteristic equation:

\[
(j\omega - 0.5)(j\omega - 1) + K j\omega = 0 \iff -\omega^2 - 1.5j\omega + 0.5 + jK\omega = 0
\]

which leads to the following two equations to solve for \( K \) and \( \omega \):

\[
\begin{cases}
-\omega^2 + 0.5 = 0 \quad \Rightarrow \quad \omega = \pm\sqrt{0.5} = \pm0.707 \\
-1.5\omega + K\omega = 0 \quad \Rightarrow \quad K = 1.5
\end{cases}
\]

Thus the closed-loop system with the above control law will be stable if \( K > 1.5 \).
PROBLEM 3 (20%)

Derive the differential equation of motion in terms of coordinate $\theta$ for the camera auto-focus mechanism shown below. The gear rotates about its center of mass and the two racks, $M_1$ and $M_2$, roll without slipping on the gear as it rotates. The coordinates you should use are defined with respect to the deformed positions of the springs.

(a) Draw the free body diagrams for the system.

The free body diagrams for this system are given below.
(b) Write down Newton and Euler’s laws for these free bodies.

Newton and Euler’s equations for the free body diagrams in part (a) are given below:

\[
J \ddot{\theta} = \tau(t) - F_1 R - F_2 R \\
M_1 \ddot{x}_1 = F_1 - K_1 x_1 - C_1 \dot{x}_1 \\
M_2 \ddot{x}_2 = F_2 - K_2 x_2
\]

(c) Derive the differential equation of motion for an input torque \( \tau(t) \) and output motion \( \theta(t) \).

After eliminating all coordinates except for the rotational coordinate by substituting the second two equations into the first and applying the two rolling constraints, \( x_1 = R \theta \) and \( x_2 = R \theta \), the following single equation of motion is obtained:

\[
J \ddot{\theta} = \tau(t) - (M_1 \ddot{x}_1 + K_1 x_1 + C_1 \dot{x}_1)R - (M_2 \ddot{x}_2 + K_2 x_2)R \\
\left( J + M_1 R^2 + M_2 R^2 \right) \ddot{\theta} + C_1 R^2 \dot{\theta} + (K_1 R^2 + K_2 R^2) \theta = \tau(t)
\]
PROBLEM 4 (30%) 

You are designing a temperature control system for the automobile passenger cabin shown in Figure 2. The differential equation of this system is given by:

\[ RCT_c' + T_c = R q_i + T_a \]

where the thermal capacitance of the cabin is \( C = 100 \, \text{J/°F} \); the thermal resistance to conduction through to the outside is \( R = 0.6 \, \text{W/°F} \); \( T_c \) is the temperature in the cabin; \( T_a \) is the outside temperature; and \( q_i \) is the heat/AC (in Watts) supplied to the cabin. The block diagram of the control system you need to design is also shown below. Answer the following questions.
(a) Show that the differential equation is as given above.

If it assumed that the temperature of the outer surface of the car is the same as the ambient temperature, then convection to the outside is zero and the following equations can be derived:

\[ \dot{E}_{\text{stored}} = Q_{\text{in}} - Q_{\text{out}} \]
\[ C \dot{T}_{c} = q_{i} - \frac{T_{c} - T_{a}}{R} \]
\[ RC \dot{T}_{c} + T_{c} = Rq_{i} + T_{a} \]

(b) For a proportional controller, \( G_c(s) = K_P \), find the closed loop transfer function between \( T_r \) and \( T_c \) and the closed loop transfer function between \( T_a \) and \( T_c \).

For \( G_c(s) = K_P \), the closed transfer functions are found using Mason’s Gain formula as follows:

\[
\frac{T_c(s)}{T_r(s)} = 0.01 \frac{K_P \frac{R}{RCs+1}}{1 + 0.01K_P \frac{R}{RCs+1}} = \frac{0.01RK_P}{RCs + 1 + 0.01RK_P}
\]
\[ = \frac{0.006K_P}{60s + 1 + 0.006K_P} \]
\[
\frac{T_c(s)}{T_a(s)} = \frac{1}{1 + 0.01K_P \frac{R}{RCs+1}} = \frac{1}{RCs + 1 + 0.006K_P}
\]
\[ = \frac{1}{60s + 1 + 0.006K_P} \]

(c) Assume the input \( T_r \) is a unit step (\( T_r = 1 \) °F) and \( T_a = 0 \) °F. If you want \( T_c \) to reach 2 % of its steady-state value in 80 sec, what value of \( K_P \) should you choose?

The system time constant determines how long it takes the cabin temperature to reach 63 % of its final steady state value. Since the closed loop transfer function between \( T_r(s) \) and \( T_c(s) \) is governed by a time constant of \( 60/(1 + 0.006K_P) \), we have to choose \( K_P \) as follows to get \( \tau = 20 \) sec:

\[
\tau = \frac{60}{1 + 0.006K_P}
\]
\[
K_P = \frac{\frac{60}{\tau} - 1}{0.006}
\]
\[ = \frac{\frac{60}{20} - 1}{0.006}
\]
\[ = 333.33 \]
(d) What is the steady-state error between $T_r$ and $T_c$ for a unit step input ($T_r = 1 ^\circ F$) for the closed loop system with proportional control and the $K_P$ value you obtained in part (c)?

The steady state error between $T_r$ and $T_c$ for a step input can be found using the final value theorem and is equal to,

$$ e_{ss} = T_{r-ss} - T_{c-ss} $$
$$ = 1^\circ F - \lim_{s\to0} sT_c(s) \frac{T_c(s)}{T_r(s)} $$
$$ = 1^\circ F - \lim_{s\to0} \frac{0.006K_P}{60s + 1 + 0.006K_P} $$
$$ = 1^\circ F - \frac{2}{1+2} $$
$$ = \frac{1^\circ}{3} F $$

(e) For a proportional integral controller, $G_c(s) = K\frac{s + \alpha}{s}$, find the closed loop transfer function between $T_r$ and $T_c$.

For $G_c(s) = K\frac{s + \alpha}{s}$, the closed transfer function is found using Mason’s Gain formula as follows:

$$ \frac{T_c(s)}{T_r(s)} = \frac{0.01}{1 + 0.01K\frac{s + \alpha}{s}} = \frac{0.01RK(s + \alpha)}{s(RC + 1) + 0.01RK(s + \alpha)} $$
$$ = \frac{0.01RK(s + \alpha)}{RCs^2 + (1 + 0.01RK)s + 0.01RK\alpha} $$
$$ = \frac{0.006K(s + \alpha)}{60s^2 + (1 + 0.006K)s + 0.006K\alpha} $$

(f) Find the steady-state error between $T_r$ and $T_c$ for a unit step input ($T_r = 1 ^\circ F$) in the closed loop system with proportional-integral control as in part (e).

The steady state error between $T_r$ and $T_c$ for a step input can be found using the final value theorem and is equal to,

$$ e_{ss} = T_{r-ss} - T_{c-ss} $$
$$ = 1^\circ F - \lim_{s\to0} sT_c(s) \frac{T_c(s)}{T_r(s)} $$
$$ = 1^\circ F - \lim_{s\to0} \frac{0.006K(s + \alpha)}{60s^2 + (1 + 0.006K)s + 0.006K\alpha} $$
$$ = 1^\circ F - \frac{0.006K\alpha}{0.006K\alpha} $$
$$ = 0^\circ F $$
PROBLEM 5 (15%)

The hydraulic lift below is used to service automobiles by positioning them vertically with the hydraulic supply pressure, $P$, and oil flow rate, $Q$. Important parameters are shown in the figure including the gravitational constant $g$, viscous damping coefficient $b$, piston area $A$ and hydraulic capacitance $C$.

Use this schematic to find the differential equation that relates the vertical position of the vehicle $y$ to the fluid flow $Q$.

(a) Draw the free body diagram of the ram-vehicle system

The free body diagram for this system is given below

where $F_r = (P - P_f)A$
(b) Write down Newton’s second law for the ram-vehicle system

\[ \sum F_y = F_r - mg - b\ddot{y} = m\dddot{y} \]

\[ m\dddot{y} + b\ddot{y} = (P - P_r)A - mg \]

(c) Write down the element law for the piston that relates \( P \) to \( Q \) and \( \dot{y} \).

\[ \dot{P} = \frac{1}{C}[Q - A\dot{y}] \]

(d) Derive the differential equation relating output \( y \) to input \( Q \).

\[ (ms^2 + bs)Y(s) = (P(s) - P_r(s))A - \frac{mg}{s} \]

\[ sP(s) = \frac{1}{C}[Q(s) - AsY(s)] \]

\[ (ms^2 + bs)Y(s) = \left( \frac{1}{Cs}[Q(s) - AsY(s)] - P_r(s) \right)A - \frac{mg}{s} \]

\[ m\dddot{y} + b\ddot{y} + \frac{A^2}{C}\dot{y} = \frac{A}{C}Q(t) - A\dot{P}_r(t) - mg\delta(t) \]
Problem 6 (10%) 

Use the frequency response data given below to answer the following questions:

(a) If the system steady-state response is given by, \( y_{ss}(t) = 20 \cos(0.6t - 0.6 \text{ rad}) \text{ cm} \), what is the input \( u(t) \)?

The response is at a frequency of 0.6 rad/s with a phase of 0.6 rad; therefore, the input must be given by:

\[
 u(t) = \frac{20}{|G(j0.6)|} \cos(0.6t - 0.6 - \angle G(j0.6)) \\
= \frac{20}{2.5} \cos(0.6t - 0.6 \text{ rad} + 80^\circ) \text{N} \\
= 8 \cos(0.6t + 0.8 \text{ rad}) \text{ N}
\]
(b) If you know that the transfer function corresponding to the frequency response data is,

\[ G(s) = \frac{A}{s^2 + 1.6s + 4} \]

find \( A \)

Since the magnitude of the frequency response function near zero frequency is 3 dB or \( \sqrt{2} \) cm/N, \( A \) must be calculated as follows:

\[
\sqrt{2} = \frac{A}{s^2 + 1.6s + 4} \bigg|_{s=j\omega=j0} = \frac{A}{4}
\]

\[ A = 4\sqrt{2} \frac{cm}{N} \]