ME 375 FINAL EXAM
Monday, December 12, 2005

Division King 10:30 / Meckl 2:30 (circle one) **Name** Solution

Instructions

(1) This is a closed book examination, but you are allowed three 8.5×11 crib sheets.

(2) You have two hours to work all six problems on the exam.

(3) Use the solution procedure we have discussed: what are you given, what are you asked to find, what are your assumptions, what is your solution, does your solution make sense. You must show all of your work to receive any credit.

(4) Circle or box-in your answers.

(5) You must write neatly and should use a logical format to solve the problems. You are encouraged to really "think" about the problems before you start to solve them. Please write your name in the top right-hand corner of each page.

(6) A table of Laplace transform pairs and properties of Laplace transforms is attached at the end of this exam set.

Problem No. 1 (40) ____________

Problem No. 2 (30) ____________

Problem No. 3 (30) ____________

Problem No. 4 (40) ____________

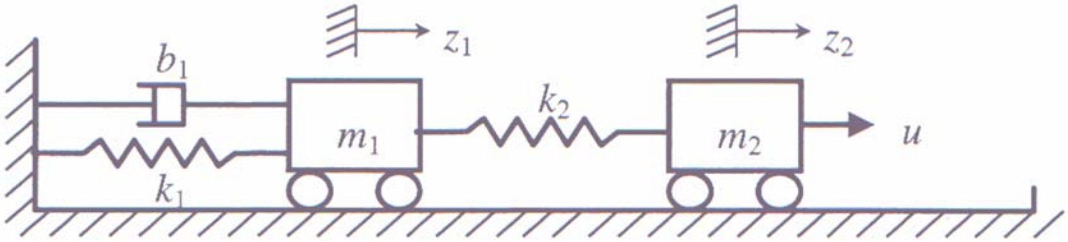
Problem No. 5 (30) ____________

Problem No. 6 (30) ____________

TOTAL (*/200)** ____________

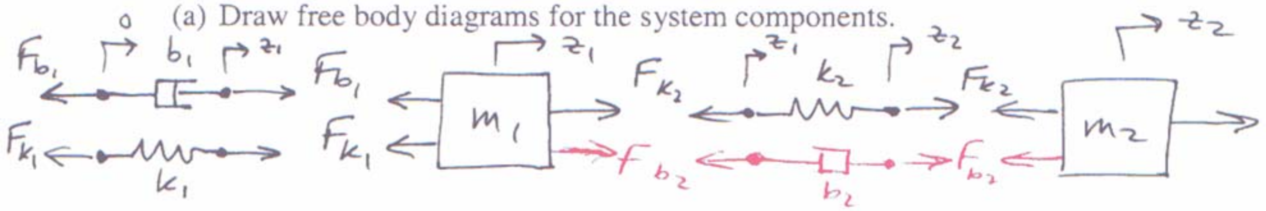
PROBLEM #1: (40 points)

Given the following translating mechanical system:



Develop the input /output model for the system shown in the figure above, where $u(t)$ is the input and $z_1(t)$ is the output by doing the following:

(a) Draw free body diagrams for the system components.



(b) Write elemental equations for each element in the system.

$F_{b_1} = b_1 \dot{z}_1 \qquad m_1 \ddot{z}_1 = F_{k_2} - F_{b_1} - F_{k_1} \qquad m_2 \ddot{z}_2 = u - F_{k_2}$

$F_{k_1} = k_1 z_1 \qquad F_{k_2} = k_2 (z_2 - z_1)$

(c) Combine these equations to determine the set of system equations of motion.

$$m_1 \ddot{z}_1 + k_2(z_1 - z_2) + b_1 \dot{z}_1 + k_1 z_1 = 0$$

$$m_2 \ddot{z}_2 + k_2(z_2 - z_1) = u$$

(d) Find the input/output model with z_1 as the output and u as the input.

$$z_2 = \frac{1}{k_2}\left(m_1 \ddot{z}_1 + b_1 \dot{z}_1 + (k_1+k_2) z_1\right)$$

$$\frac{m_2}{k_2}\left[m_1 \ddddot{z}_1 + b_1 \dddot{z}_1 + (k_1+k_2)\ddot{z}_1\right] + m_1 \ddot{z}_1 + b_1 \dot{z}_1 + (k_1+\cancel{k_2}) z_1 - \cancel{k_2 z_1} = u$$

$$\boxed{m_1 m_2 \ddddot{z}_1 + m_2 b_1 \dddot{z}_1 + (m_2 k_1 + m_2 k_2 + m_1 k_2)\ddot{z}_1 + b_1 k_2 \dot{z}_1 + k_1 k_2 z_1 = k_2 u}$$

(e) Now add an additional damper b_2 in parallel with k_2 connecting m_1 to m_2 and again find the input/output model with z_1 as the output and u as the input.

$$m_1 \ddot{z}_1 = F_{k_2} - F_b - F_{k_1} + F_{b_2} \qquad m_2 \ddot{z}_2 = u - F_{k_2} - F_{b_2}$$

$$F_{b_1} = b_1 \dot{z}_1 \qquad F_{k_2} = k_2(z_2 - z_1)$$

$$F_{k_1} = k_1 z_1 \qquad F_{b_2} = b_2(\dot{z}_2 - \dot{z}_1)$$

$$m_1 \ddot{z}_1 + k_2(z_1 - z_2) + b_1 \dot{z}_1 + k_1 z_1 + b_2(\dot{z}_1 - \dot{z}_2) = 0$$

$$m_2 \ddot{z}_2 + k_2(z_2 - z_1) + b_2(\dot{z}_2 - \dot{z}_1) = u$$

$$\left[m_1 s^2 + (b_1+b_2)s + (k_1+k_2)\right] Z_1(s) - (b_2 s + k_2) Z_2(s) = 0$$

$$-(b_2 s + k_2) Z_1(s) + (m_2 s^2 + b_2 s + k_2) Z_2(s) = U(s)$$

$$\begin{bmatrix} m_1 s^2 + (b_1+b_2)s + (k_1+k_2) & -(b_2 s + k_2) \\ -(b_2 s + k_2) & m_2 s^2 + b_2 s + k_2 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ U \end{bmatrix}$$

$$Z_1(s) = \frac{\begin{bmatrix} 0 & -(b_2 s + k_2) \\ U & m_2 s^2 + b_2 s + k_2 \end{bmatrix}}{\left[m_1 s^2 + (b_1+b_2)s + (k_1+k_2)\right]\left[m_2 s^2 + b_2 s + k_2\right] - (b_2 s + k_2)^2}$$

$$\boxed{\frac{Z_1(s)}{U(s)} = \frac{b_2 s + k_2}{\left[m_1 s^2 + (b_1+b_2)s + (k_1+k_2)\right]\left[m_2 s^2 + b_2 s + k_2\right] - (b_2 s + k_2)^2}}$$

PROBLEM #2: (30 points)

The transfer function for a mechanical system is given by:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{3}{s^2 + 4s + 3}$$

(a) Write down expressions for $G(j\omega)$, $|G(j\omega)|$, and $\angle G(j\omega)$.

$$G(j\omega) = \frac{3}{(3-\omega^2) + 4j\omega}$$

$$|G(j\omega)| = \frac{3}{\sqrt{(3-\omega^2)^2 + (4\omega)^2}}$$

$$\angle G(j\omega) = -\tan^{-1} \frac{4\omega}{3-\omega^2}$$

(b) Find the magnitude and phase of $G(j\omega)$ at $\omega = 1$ and 5 rad/sec.

$|G(j1)| = 0.67$, $\angle G(j1) = -63.4°$

$|G(j5)| = 0.1$, $\angle G(j5) = -137.7°$

(c) Find the steady-state output $y_{ss}(t)$ of the system to an input of $u(t) = 5 + \sin(t) + 3\sin(5t)$.
(Note: collect terms but leave as a function of t.)

$$y_{ss}(t) = 5 + 0.67 \sin(t - 63.4°) + 0.3 \sin(5t - 137.7°)$$

PROBLEM #3: (30 points)

Given the following input/output model:

$$2\ddot{x} + 8\dot{x} + 6x = 6u(t), \qquad x(0) = 0, \qquad \dot{x}(0) = 2$$

where $u(t)$ is the input (not specified).

(a) Write the solution $X(s)$ as a sum of two terms, one the *forced* response, the other the *free* response. The total solution should be a function of $U(s)$.

$$s^2X(s) - sx(0) - \dot{x}(0) + 4\left[sX(s) - x(0)\right] + 3X(s) = 3U(s)$$

$$(s^2 + 4s + 3)X(s) - 2 = 3U(s)$$

$$X(s) = \underbrace{\frac{2}{s^2+4s+3}}_{\text{free}} + \underbrace{\frac{3}{s^2+4s+3}U(s)}_{\text{forced}}$$

(b) Generate the transfer function that relates $X(s)$ to $U(s)$.

$$G(s) = \frac{X(s)}{U(s)} = \frac{3}{s^2+4s+3} = \frac{3}{(s+1)(s+3)}$$

(c) Find the poles and zeros of the transfer function in part (b).

zeros: none

poles: $-1, -3$

(d) Find the static gain of the transfer function in part (b).

$$G(0) = 1$$

(e) Generate an expression for the free response $x(t)$ when $u(t) = 0$.

$$X(s) = \frac{2}{(s+1)(s+3)} = \frac{1}{s+1} + \frac{-1}{s+3}$$

$$x(t) = e^{-t} - e^{-3t}$$

(f) Based on your solution in part (e), please explain how the poles of the transfer function you found in part (c) are related to the free response.

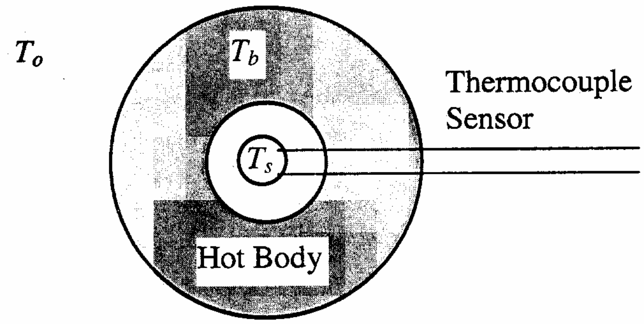
Poles determine the exponent coefficients of t.

(g) Is this system stable? Please explain.

Yes, both poles are negative real.

PROBLEM #4: **(40 points)**

A thermocouple sensor is used to measure the temperature T_b of a hot body that is being heated inside an oven. The oven temperature is given by T_o. The temperature of the thermocouple mass is given by T_s. A schematic of this process is shown below:



The specific heat of the hot body is denoted by c_b with mass m_b. The specific heat of the thermocouple is denoted by c_s with mass m_s. The convective heat transfer coefficient between the oven and the hot body is given by h_b, with surface area given by A_b. The convective heat transfer coefficient between the hot body and the thermocouple is given by h_s, with surface area given by A_s.

(a) Write down energy balances for the hot body and the thermocouple.

$$m_b c_b \frac{dT_b}{dt} = h_b A_b (T_o - T_b) - h_s A_s (T_b - T_s)$$

$$m_s c_s \frac{dT_s}{dt} = h_s A_s (T_b - T_s)$$

(b) Develop the input-output equation relating the thermocouple temperature to the oven temperature.

$$T_b = \frac{1}{h_s A_s}\left[m_s c_s \frac{dT_s}{dt}\right] + T_s$$

$$\frac{m_b c_b}{h_s A_s} m_s c_s \frac{d^2T_s}{dt^2} + m_b c_b \frac{dT_s}{dt} + (h_b A_b + h_s A_s)\left(\frac{m_s c_s}{h_s A_s}\frac{dT_s}{dt} + T_s\right)$$

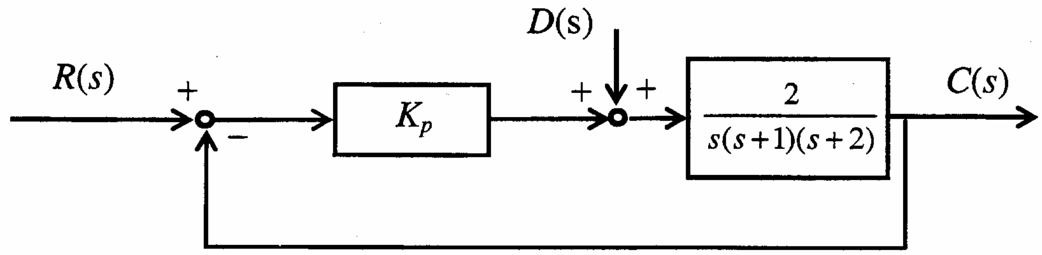
$$- h_s A_s T_s = h_b A_b T_o$$

(c) Does the thermocouple measurement correctly track the hot body temperature over time? Please explain.

No, since some of the energy must exit the hot body in order to register on the thermocouple.

PROBLEM #5: (30 points)

For the system given in the following figure:



(a) Find the transfer function $C(s)/R(s)$ (assume $D(s) = 0$).

$$\frac{C(s)}{R(s)} = \frac{2K_p}{s(s+1)(s+2) + 2K_p}$$

(b) Find the transfer function $C(s)/D(s)$ (assume $R(s) = 0$).

$$\frac{C(s)}{D(s)} = \frac{2}{s(s+1)(s+2) + 2K_p}$$

(c) Find the gain K_p for which the roots of the characteristic equation are purely imaginary. (Hint: set $s = j\omega$.)

$$s^3 + 3s^2 + 2s + 2K_p = 0$$

$$-j\omega^3 - 3\omega^2 + 2j\omega + 2K_p = 0$$

$$(2K_p - 3\omega^2) + j\underbrace{(2\omega - \omega^3)}_{\omega(2-\omega^2) = 0} = 0 \quad \Rightarrow \omega^2 = 2$$

$$2K_p = 3\omega^2 = 6 \quad \Rightarrow \boxed{K_p = 3}$$

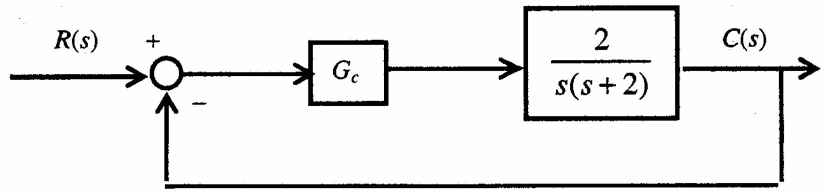
(d) Determine the range of gain K_p for which the closed-loop system is stable.

$$0 < K_p < 3$$

since roots cross over into positive real territory for $K_p > 3$.

PROBLEM #6: (30 points)

Given the following feedback control system with controller G_c:



Assume that a PD controller is used, i.e., $G_c(s) = K_p + K_d s$.

(a) Determine the closed-loop transfer function $C(s)/R(s)$ as function of K_p and K_d.

$$\frac{C(s)}{R(s)} = \frac{2(K_p + K_d s)}{s(s+2) + 2(K_p + K_d s)}$$

(b) Write the characteristic equation in standard second-order form and write expressions for K_p and K_d as functions of damping ratio and natural frequency.

$$s^2 + \underbrace{(2 + 2K_d)}_{2\zeta\omega_n} s + \underbrace{2K_p}_{\omega_n^2} = 0$$

$$\boxed{\begin{aligned} K_p &= \tfrac{1}{2}\omega_n^2 \\ K_d &= \zeta\omega_n - 1 \end{aligned}}$$

(c) Find values of K_p and K_d that yield a 2% settling time of 1 second and 16% overshoot for a step reference input.

$16\%OS \Rightarrow \zeta = 0.5$ since $\%OS = 100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}}$

$$t_s\Big|_{2\%} = \frac{4}{\zeta\omega_n} = 1 \Rightarrow \zeta\omega_n = 4$$

$$\Rightarrow 2\zeta\omega_n = 8 \Rightarrow \omega_n = 8$$

$$\boxed{\begin{aligned} K_p &= 32 \\ K_d &= 3 \end{aligned}}$$