ME 375 FINAL EXAM
Tuesday, Dec. 15, 2009

Division Meckl 10:30 / Deng 2:30 (circle one) **Name** Solution

Instructions

(1) This is a closed book examination, but you are allowed three single-sided 8.5”×11” crib sheets.

(2) You have two hours to work all ~~five~~ six problems on the exam.

(3) Use the solution procedure we have discussed: what are you given, what are you asked to find, what are your assumptions, what is your solution, does your solution make sense. You must show all of your work to receive any credit.

(4) Circle or box-in your answers.

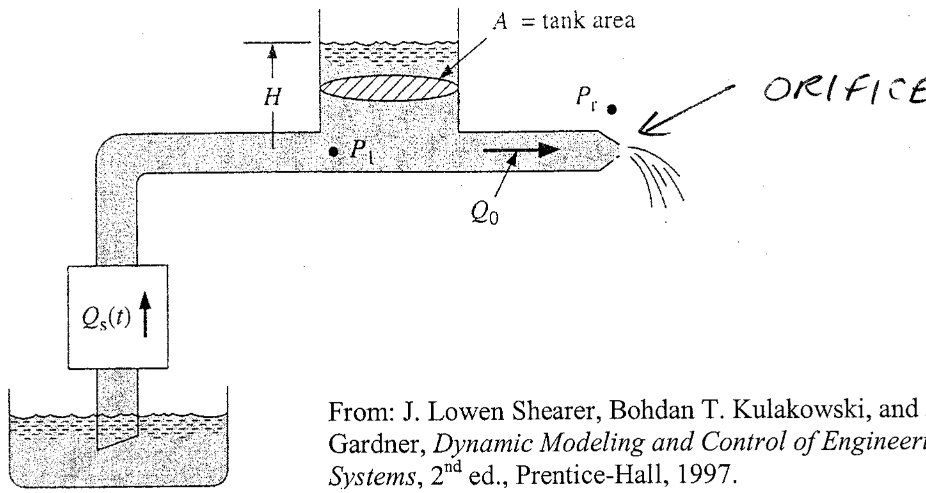
(5) You must write neatly and should use a logical format to solve the problems. You are encouraged to really “think” about the problems before you start to solve them. Please write your name in the top right-hand corner of each page.

(6) A table of Laplace transform pairs and properties of Laplace transforms is attached at the end of this exam set.

Problem No. 1 (18)	__________
Problem No. 2 (15)	__________
Problem No. 3 (15)	__________
Problem No. 4 (20)	__________
Problem No. 5 (12)	__________
Problem No. 6 (20)	__________
TOTAL (*/100)**	__________
BONUS (*/5)**	__________
TOTAL (*/100)**	__________

PROBLEM 1: (18%)

You have been asked to develop the differential equation that describes the following hydraulic circuit:

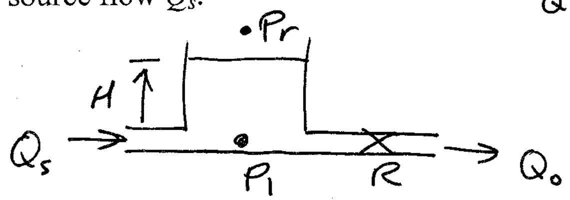


From: J. Lowen Shearer, Bohdan T. Kulakowski, and John F. Gardner, *Dynamic Modeling and Control of Engineering Systems*, 2nd ed., Prentice-Hall, 1997.

In this system, the variable flow source Q_s is used to maintain the level of the tank, which supplies flow through the orifice. You should treat the orifice as a fluid resistance R. The fluid has density ρ and the entire system is subjected to gravity with acceleration g.

(a) (8%) Determine the differential equation that describes the height H in terms of the source flow Q_s.

Q = volumetric flow rate



$$P_{1r} = \rho g H = R Q_o, \quad Q_s - Q_o = A\dot{H}$$

$$Q_o = \frac{\rho g}{R} H$$

$$A\dot{H} + \frac{\rho g}{R} H = Q_s$$

$$\boxed{\dot{H} + \frac{\rho g}{A}\frac{1}{R} H = \frac{1}{A} Q_s}$$

PROBLEM 1: (cont)

(b) (5%) Find an expression for the time constant for this first-order system.

$$\underbrace{\frac{A}{\rho g} R}_{\tau} \dot{H} + H = \frac{1}{\rho g} R Q_s$$

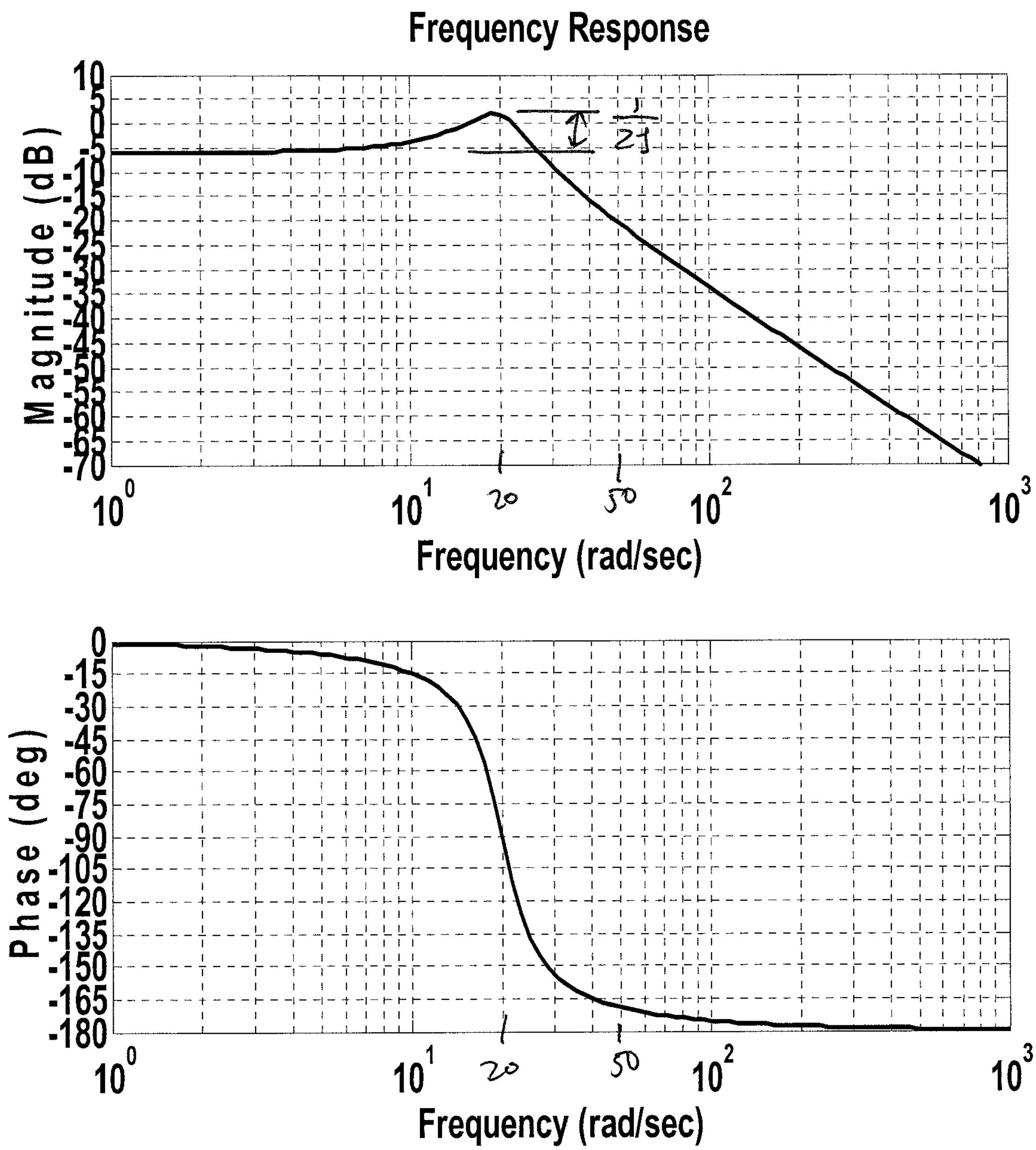
$$\boxed{\tau = \frac{A}{\rho g} R}$$

(c) (5%) Determine an expression for the steady-state height of the tank H_{ss} in terms of the source flow Q_s and the parameters defined above.

$$\boxed{H_{ss} = \frac{1}{\rho g} R Q_s}$$

PROBLEM 2: (15%)

The frequency response for an electric circuit is given in the Bode plots below:



(a) (4%) Determine the natural frequency, damping ratio, and static gain for this system.

$\boxed{\omega_n = 20 \text{ r/s}}$, where phase $= -90°$.

$\boxed{K_s = -6 \text{ dB} = 0.5}$

$\frac{1}{2\zeta} \approx 8 \text{ dB} = 2.5 \Rightarrow \boxed{\zeta = 0.2}$

PROBLEM 2: (cont)

(b) (3%) Generate an expression for this system transfer function $G(s)$, which represents the ratio of output Y(s) to input U(s).

$$G(s) = \frac{0.5\,(20)^2}{s^2 + 8s + 20^2}$$

$$\boxed{G(s) = \frac{200}{s^2 + 8s + 400}}$$

(c) (4%) Find the magnitude and phase of $G(j\omega)$ at $\omega = 20$ and 50 rad/sec.

From plot :

$$|G(j20)| = 2\,dB = 1.26 \quad , \quad \angle G(j20) = -90^\circ$$

$$|G(j50)| = -20\,dB = 0.1 \quad , \quad \angle G(j50) = -170^\circ$$

(d) (4%) Find the input $u(t)$ that would generate the steady-state output of $y_{ss}(t) = 3 + \sin(20t) + 2\sin(50t)$. Let input $u(t) = A_0 + A_1 \sin(20t + \phi_1) + A_2 \sin(50t + \phi_2)$

$$y_{ss}(t) = A_0\, G(0) + A_1 |G(j20)| \sin(20t + \phi_1 + \angle G(j20)) + A_2 |G(j50)| \sin(50t + \phi_2 + \angle G(j50))$$

$$\therefore \quad A_0\, G(0) = 3 \Rightarrow A_0 = 6$$

$$A_1 |G(j20)| = 1 \Rightarrow A_1 = \frac{1}{1.26} = 0.8$$

$$\phi_1 + \angle G(j20) = 0 \Rightarrow \phi_1 = +90^\circ$$

$$A_2 |G(j50)| = 2 \Rightarrow A_2 = \frac{2}{0.1} = 20$$

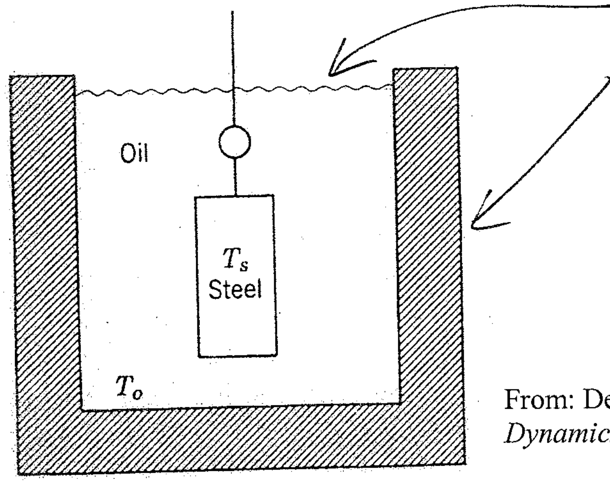
$$\phi_2 + \angle G(j50) = 0 \Rightarrow \phi_2 = +170^\circ$$

$$\boxed{u(t) = 6 + 0.8 \sin(20t + 90^\circ) + 20 \sin(50t + 170^\circ)}$$

PROBLEM 3: (15+5%)

The diagram below shows a piece of hot steel being quenched in a vat of oil:

NO HEAT TRANSFER



From: Dean Karnopp and Ronald Rosenberg, *System Dynamics: A Unified Approach*, Wiley, 1975.

The steel has a thermal capacitance C_s and the heat transfer between the steel and oil is given by thermal resistance R.

(a) (5%) Assuming that the vat is large so that its temperature T_o stays constant over time, derive a differential equation that describes the temperature response of the steel $T_s(t)$.

$$C_s \frac{dT_s}{dt} = 0 - \frac{1}{R}(T_s - T_o)$$

$$C_s \frac{dT_s}{dt} + \frac{1}{R} T_s = \frac{1}{R} T_o$$

$$\boxed{R C_s \frac{dT_s}{dt} + T_s = T_o}.$$

PROBLEM 3: (cont)

(b) (10%) Redo part (a), but this time assume that the vat temperature will change. The corresponding thermal capacitance of the oil is given by C_o.

$$\boxed{RC_s \frac{dT_s}{dt} + T_s = T_o}$$

$$C_o \frac{dT_o}{dt} = \frac{1}{R}(T_s - T_o)$$

$$\boxed{RC_o \frac{dT_o}{dt} + T_o = T_s}$$

$$RC_o \frac{d}{dt}\left[RC_s\dot{T}_s + T_s\right] + RC_s\dot{T}_s + T_s = T_s$$

$$\boxed{RC_oRC_s\ddot{T}_s + (RC_o + RC_s)\dot{T}_s = 0}$$

(c) (5%) BONUS: determine the final temperature of the steel in the conditions for part (b).

Let $T_s(0) = T_{so}$ and $T_o(0) = T_{oo}$

$$\frac{dT_s(0)}{dt} = \frac{1}{RC_s}(T_{oo} - T_{so})$$

$\int dt$

$$RC_oRC_s\dot{T}_s + (RC_o + RC_s)T_s = K$$

plug in $t = 0$:

$$\frac{RC_o\cancel{RC_s}}{\cancel{RC_s}}(T_{oo} \cancel{- T_{so}}) + (\cancel{RC_o} + RC_s)T_{so} = K$$

$$K = RC_oT_{oo} + RC_sT_{so}$$

at s.s.: $(RC_o + RC_s)T_s = K = RC_oT_{oo} + RC_sT_{so}$

$$\Rightarrow \boxed{T_s = \frac{C_o}{C_o + C_s}T_{oo} + \frac{C_s}{C_o + C_s}T_{so}}$$

PROBLEM 4: (20%):

Consider the feedback system in Figure 4, where the plant transfer function is

$$G(s) = \frac{1}{s^2 + a_1 s + a_2}$$

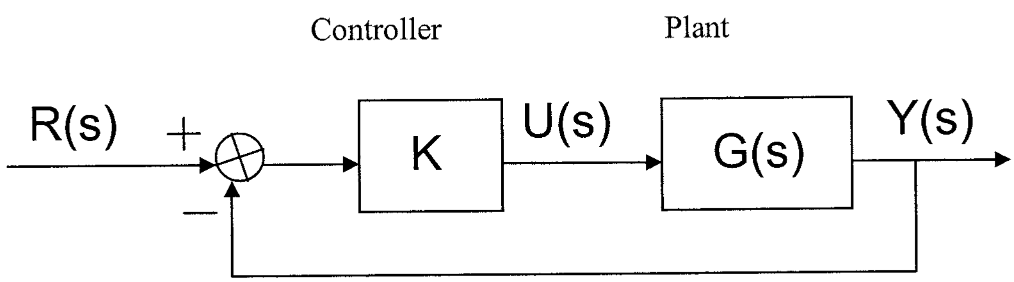


Figure 4. Feedback System

(a) (5%). Obtain the closed-loop transfer function $\frac{Y(s)}{R(s)}$.

$$\boxed{\frac{Y(s)}{R(s)} = \frac{K}{s^2 + a_1 s + a_2 + K}}$$

PROBLEM 4: (cont)

(b) (15%). Figure 5 shows the unit step input response of the output, y(t), of the closed-loop system in Figure 4, (i.e. $R(s) = \frac{1}{s}$, $y(0) = \dot{y}(0) = 0$.)

Calculate the values of the control gain, K, and of the plant parameters a_1 and a_2. You need only one decimal point accuracy in your calculation.

(Hint: Consider time domain specifications, also notice that $y_{ss} = \lim_{t\to\infty} y(t) = 1$.)

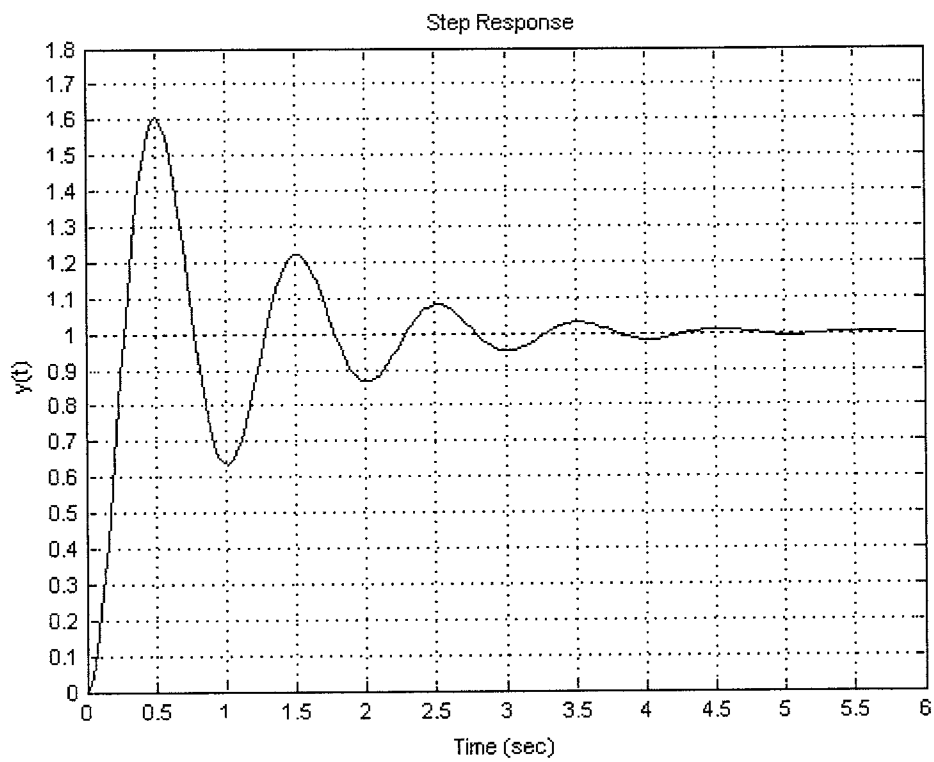


Figure 5. Closed-loop System Unit Step Response

at ss: $\frac{K}{a_2+K} = 1 \Rightarrow \boxed{a_2 = 0}$

$\%OS = 60\% = 100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}} \Rightarrow 0.6 = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$

$\ln 0.6 = -\pi\zeta/\sqrt{1-\zeta^2}$

$(1-\zeta^2)(\ln 0.6)^2 = \pi^2\zeta^2$

$[\pi^2 + (\ln 0.6)^2]\,\zeta^2 = (\ln 0.6)^2$

$\Rightarrow \zeta^2 = 0.026$

$\Rightarrow \boxed{\zeta = 0.16}$

$\omega_d = 1\ \text{sec} = \omega_n\sqrt{1-\zeta^2}$

$\boxed{\omega_n = \frac{1}{\sqrt{1-\zeta^2}} = 0.99\ r/s.}$

$\boxed{a_1 = 2\zeta\omega_n = 0.3}$

$\boxed{K = \omega_n^2 = 1}$

PROBLEM 5 (12%):

For the feedback system shown in Figure 6, $G(s) = \dfrac{1}{s(s+2)(s+4)}$. The root locus for $K \in [0, \infty)$ is sketched in Figure 7.

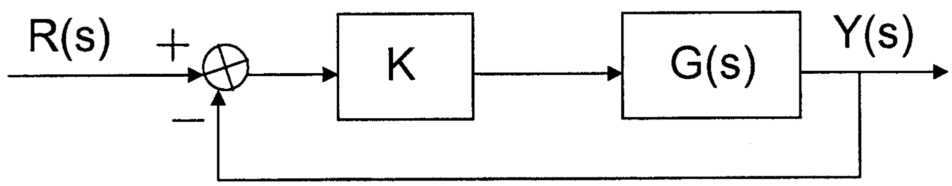


Figure 6. Feedback System

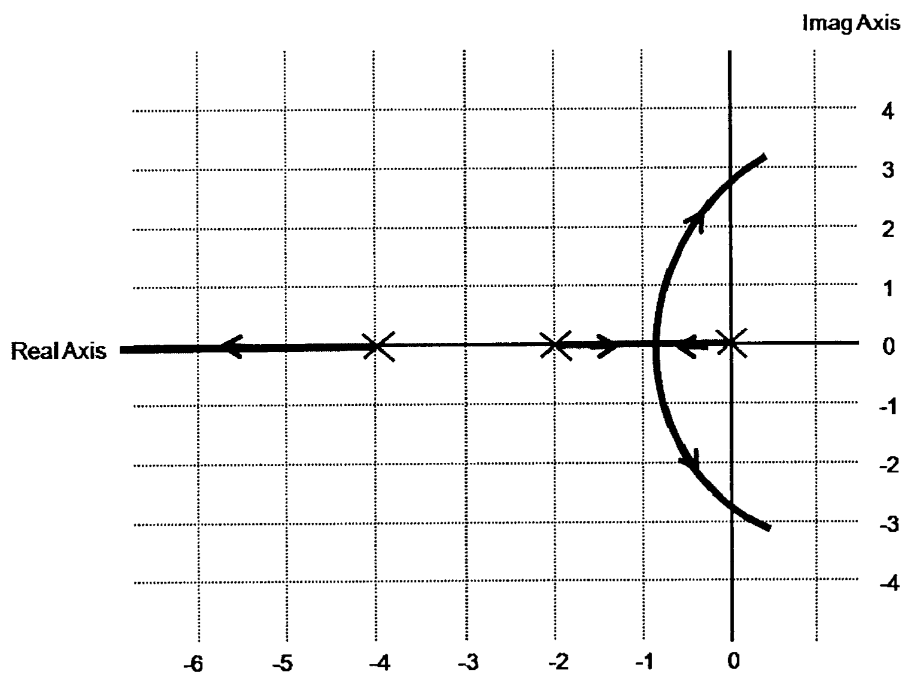


Figure 7. Root locus

(a) (6%) Find the range of K such that the closed loop system is stable.

$$s^3 + 6s^2 + 8s + K = 0$$

$$-j\omega^3 - 6\omega^2 + 8j\omega + K = 0$$

$$(K - 6\omega^2) + j\,\underbrace{(8\omega - \omega^3)}_{} = 0$$

$\downarrow$

$\omega = 0$

$\omega^2 = 8$

$K = 6\omega^2$

$= 48$

range: $\boxed{0 < K < 48}$

PROBLEM 5: (cont)

(b) (6%) Find the value of K such that the closed loop system has the smallest settling time.

char. eq: $1 + K \dfrac{1}{s(s+2)(s+4)} = 0$

$\Rightarrow \quad K = -s(s+2)(s+4) = -s^3 - 6s^2 - 8s$

$\dfrac{dK}{ds} = -3s^2 - 12s - 8 = 0$

$3s^2 + 12s + 8 = 0$

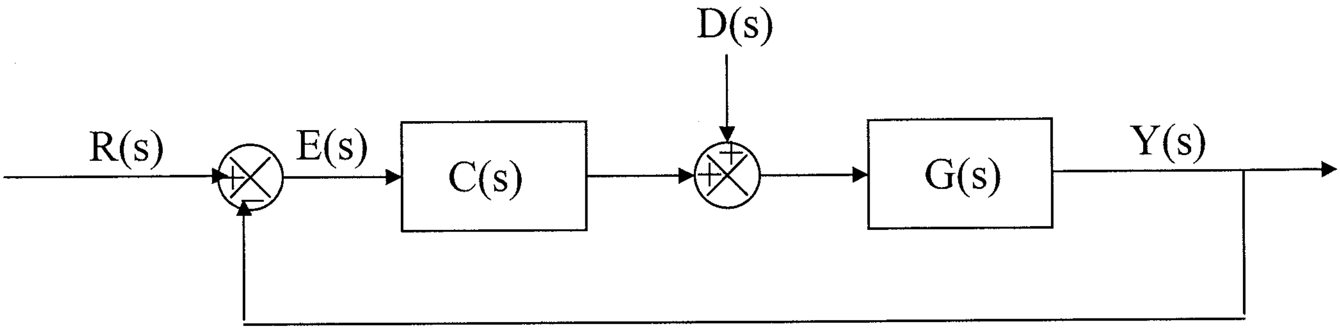
$s = -\dfrac{12}{3 \cdot 2} \pm \dfrac{1}{3 \cdot 2} \sqrt{12^2 - 4 \cdot 24}$

$s = \begin{cases} -0.845 \leftarrow \\ -3.155 \end{cases}$

$\boxed{K = 3}$

PROBLEM 6: (20%):

Consider the system shown below:



The plant transfer function is given by

$$G(s) = \frac{1}{s(10s+3)}$$

The controller is PD (proportional plus derivative) and is given by

$$C(s) = K_P + K_D s$$

(a) (6%) Write the transfer functions (a) from input R(s) to the error E(s); and (b) from disturbance D(s) to the error E(s).

$$Y(s) = G(s)\left[D(s) + C(s)E(s)\right]$$

$$E(s) = R(s) - Y(s)$$

$$E(s) = R(s) - G(s)\left[D(s) + C(s)E(s)\right]$$

$$\left[1 + G(s)C(s)\right]E(s) = R(s) - G(s)D(s)$$

$$E(s) = \frac{1}{1+GC}R(s) - \frac{G}{1+GC}D(s)$$

$$\frac{E(s)}{R(s)} = \frac{1}{1 + \frac{K_p + K_D s}{s(10s+3)}} = \frac{s(10s+3)}{s(10s+3) + K_p + K_D s}$$

$$\frac{E(s)}{D(s)} = -\frac{\frac{1}{s(10s+3)}}{1 + \frac{K_p + K_D s}{s(10s+3)}} = \frac{-1}{s(10s+3) + K_p + K_D s}$$

PROBLEM 6: (cont)

(b) (7%) The performance specifications require the steady state error due to a unit-step input ($R(s) = \frac{1}{s}$) to be zero and the steady-state error due to a unit-step disturbance ($D(s) = \frac{1}{s}$) to be no greater than 0.1 in magnitude. What values of gains K_P and K_D will satisfy these conditions?

$$E(s)\Big|_R = \frac{s(10s+3)}{s(10s+3) + K_p + K_D s} \cdot \frac{1}{s}$$

$$e_{ss}\Big|_R = \lim_{s\to 0} sE(s)\Big|_R = 0 \quad \checkmark$$

$$E(s)\Big|_D = \frac{-1}{s(10s+3)+K_p+K_D s} \cdot \frac{1}{s}$$

$$\left| e_{ss}\Big|_D \right| = \left| \lim_{s\to 0} sE(s)\Big|_D \right| = +\frac{1}{K_p} \le 0.1$$

$$\Rightarrow \boxed{K_p \ge 10}$$

$\boxed{K_D \text{ can be anything}}$

(c) (7%). The performance specifications require that the time constant $\tau = 1$ and the damping ratio $\zeta = 0.707$. Compute the values of the gains K_P and K_D that satisfy these conditions.

char eq: $s(10s+3) + K_p + K_D s = 0$

$$10s^2 + (3+K_D)s + K_p = 0$$

$$s^2 + \underbrace{\frac{3+K_D}{10}}_{2\zeta\omega_n} s + \underbrace{\frac{K_p}{10}}_{\omega_n^2} = 0$$

$$\tau = \frac{1}{\zeta\omega_n} = 1 \Rightarrow \omega_n = \frac{1}{\zeta} = \frac{1}{\sqrt{2}/2} = \sqrt{2}$$

$$\frac{K_p}{10} = 2 \Rightarrow \boxed{K_p = 20}$$

$$\frac{3+K_D}{10} = 2 \Rightarrow \boxed{K_D = 17}$$