ME 375 FINAL EXAM
Monday, May 3, 2004

Division Meckl 10:30 / King 1:30 (circle one) **Name** Solution

Instructions

(1) This is a closed book examination, but you are allowed three 8.5×11 crib sheets.

(2) You have two hours to work all the problems on the exam.

(3) Use the solution procedure we have discussed: what are you given, what are you asked to find, what are your assumptions, what is your solution, does your solution make sense. You must show all of your work to receive any credit.

(4) You must write neatly and should use a logical format to solve the problems. You are encouraged to really "think" about the problems before you start to solve them. Please write your name in the top right-hand corner of each page.

(5) A table of Laplace transform pairs and properties of Laplace transforms is attached at the end of this exam set.

Section A Problems	Section A Score	Section B Problems	Section B Score
1. (10)		1. (20)	
2. (8)		2. (20)	
3. (5)		3. (30)	
4. (5)		4. (30)	
5. (6)			
6. (10)			
7. (12)			
8. (6)			
9. (10)			
10. (8)			
11. (6)			
12. (14)			
Total (A)	/100	Total (B)	/100
		Total (A+B)	/200

A.1 (10%) Using Laplace transforms, find the total response $y(t)$ of a system with equation of motion given by:

$$\ddot{y} + 6\dot{y} + 13y = 13u$$

when the input $u(t)$ is a unit step, $y(0) = 0$, and $\dot{y}(0) = 0$.

$$(s^2 + 6s + 13)\,Y(s) = 13\,U(s) = \frac{13}{s}$$

$$Y(s) = \frac{13}{s(s^2+6s+13)} = \frac{A}{s} + \frac{Bs + C}{s^2+6s+13}$$

$$= \frac{As^2 + 6As + 13A + Bs^2 + Cs}{s(s^2+6s+13)}$$

s^2: $A + B = 0 \Rightarrow B = -1$

s: $6A + C = 0 \Rightarrow C = -6$

1: $13A = 13 \Rightarrow A = 1$

$$\Rightarrow Y(s) = \frac{1}{s} - \frac{s+6}{s^2+6s+13} = \frac{1}{s} - \frac{s+6}{(s+3)^2 + 2^2}$$

$$= \frac{1}{s} - \frac{s+3}{(s+3)^2+2^2} - \frac{\frac{3}{2}\,2}{(s+3)^2+2^2}$$

$$\boxed{y(t) = 1 - e^{-3t}\cos 2t - \tfrac{3}{2}e^{-3t}\sin 2t}$$

A.2 **(8%)** A two-degree-of-freedom mechanical system is described by the following two differential equations:

$$\ddot{x}_1 + 3\dot{x}_1 - 2\dot{x}_2 + 5x_1 - 5x_2 = 0$$
$$4\ddot{x}_2 - 2\dot{x}_1 + 3\dot{x}_2 - 5x_1 + 5x_2 = F$$

Assuming that the state variables are given by $q_1 = x_1$, $q_2 = \dot{x}_1$, $q_3 = x_2$, $q_4 = \dot{x}_2$, the input is F, and the output is x_2, write down the state and output equations for this system.

$$\dot{q}_1 = q_2$$
$$\dot{q}_2 = -5q_1 - 3q_2 + 5q_3 + 2q_4$$
$$\dot{q}_3 = q_4$$
$$\dot{q}_4 = \frac{5}{4}q_1 + \frac{2}{4}q_2 - \frac{5}{4}q_3 - \frac{3}{4}q_4 + \frac{1}{4}F$$

$$y = q_3$$

A.3 **(5%)** The motion of a system is governed by $\ddot{y} + 4y = 3u$, where y is the output of the system, and u is the input of the system. Get the transfer function from input to output. Find the poles of the transfer function. Determine the stability of this system with a brief explanation.

$$(s^2 + 4)\,Y(s) = 3\,U(s)$$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{3}{s^2+4}}$$

poles @ $s = \pm j2$

not stable ; poles do NOT have neg. real part

A.4 (5%) A second order system is described by $\ddot{y}+3\dot{y}+36y=36u$, where y is the output of the system, and u is the input of the system. Find the natural frequency, damping ratio, and static gain of this system.

$$s^3 + \underbrace{3s}_{2\zeta\omega_n} + \underbrace{36}_{\omega_n^2} = 0$$

$$\boxed{\begin{aligned} \omega_n &= 6 \ r/s \\ \zeta &= \frac{3}{12} = 0.25 \\ K_s &= \frac{36}{36} = 1 \end{aligned}}$$

A.5 (6%) You have been given the following parameters for a dc motor:

$$R_A = 0.38 \text{ ohm}$$
$$K_T = 0.05 \text{ Nm/amp}$$
$$K_b = 0.05 \text{ V/(rad/sec)}$$
$$B = 2.9\times10^{-4} \text{ Nm/(rad/sec)}$$

The steady-state torque-speed relationship is given by the following formula:

$$T_L = \frac{K_T}{R_A}e_i - \frac{R_A B + K_b K_T}{R_A}\omega_{ss}$$

where e_i represents the nominal input voltage.

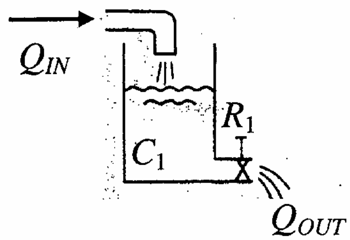
(a) Compute the stall torque (where speed is zero) when the nominal input voltage is 15 volts.

$$T_L = 1.97 \text{ Nm.}$$

(b) Compute the no-load speed when the nominal input voltage is 15 volts.

$$\omega_{ss} = \frac{K_T}{R_A B + K_b K_T} e_i = 287.3 \text{ rad/sec.}$$

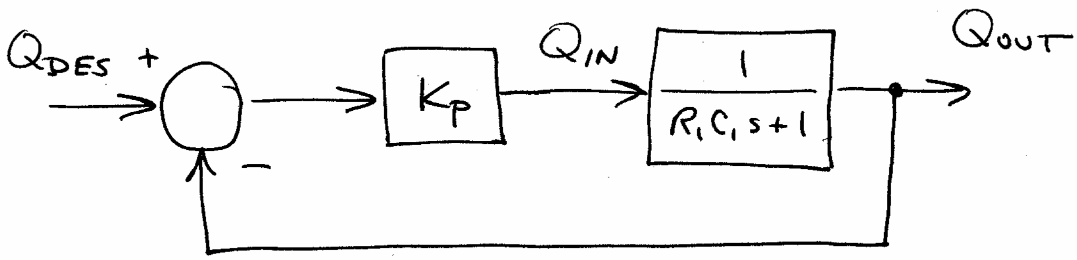
A.6 (10%) Consider the flow control system shown below:



$$\frac{1}{C_1}\int (Q_{IN} - Q_{OUT})\,dt = R_1 Q_{OUT}$$

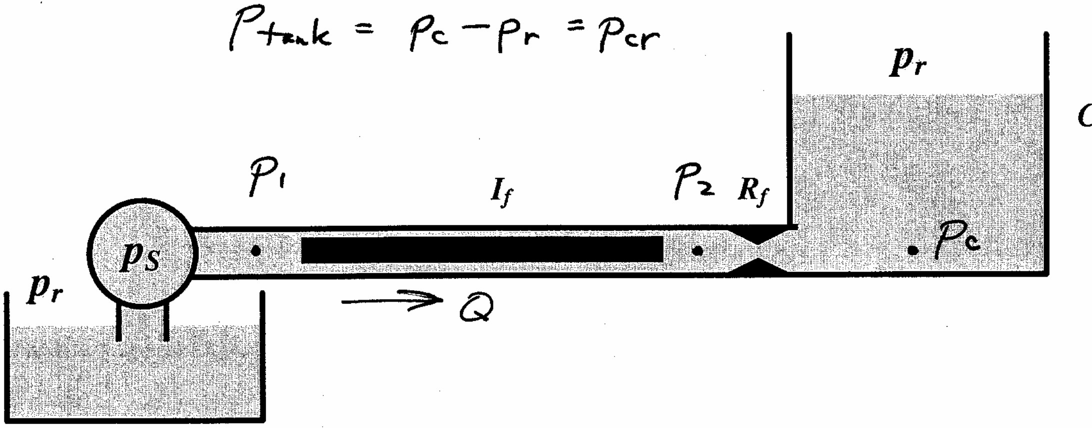
$$R_1 C_1 \dot{Q}_{OUT} + Q_{OUT} = Q_{IN}$$

The tank serves to smooth out the outlet flow rate and provides back-up flow in case the inlet flow is interrupted. However, because of the tank dynamics, the inlet flow must be properly controlled to maintain desired outlet flow. Thus, a flow sensor is used to measure the outlet flow rate Q_{OUT}. This value is compared with the desired flow rate Q_{DES} and the difference is used in a proportional controller to set the proper valve opening to regulate the inlet flow Q_{IN}. Generate a block diagram that describes this flow control system.



A.7 (12%) Consider the fluid system shown below. Derive the differential equation relating the gage pressure at the bottom of the tank p_{tank} to the pump gage pressure p_S.

$$P_{tank} = P_c - P_r = P_{cr}$$



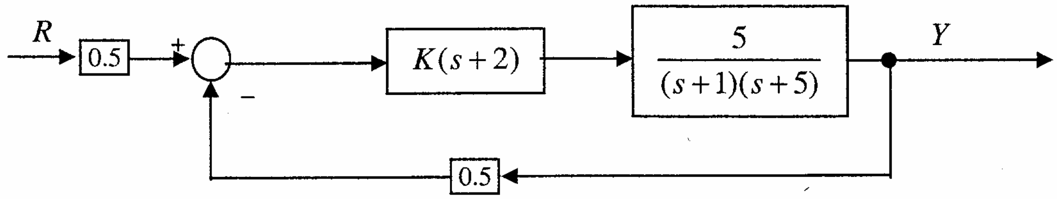
$$P_{r1} + P_{12} + P_{2c} + P_{cr} = 0 \qquad P_{r1} = -P_{1r} = -P_s$$

$$P_{12} = I_f \frac{dQ}{dt} \qquad P_{cr} = \frac{1}{C_f}\int Q\,dt = P_{tank}$$

$$P_{2c} = R_f Q \qquad \Rightarrow Q = C_f \dot{P}_{tank}$$

$$\Rightarrow \boxed{I_f C_f \ddot{P}_{tank} + R_f C_f \dot{P}_{tank} + P_{tank} = P_s}$$

A.8 **(6%)** Obtain the transfer function $G_{CL}(s) = \dfrac{Y(s)}{R(s)}$ for the system described by the following block diagram. Express it as function of *K*.



$$G_{CL}(s) = \frac{Y(s)}{R(s)} = \frac{0.5\,\dfrac{5K(s+2)}{(s+1)(s+5)}}{1 + 0.5\,\dfrac{5K(s+2)}{(s+1)(s+5)}}$$

$$\boxed{G_{CL}(s) = \frac{2.5K(s+2)}{(s+1)(s+5) + 2.5K(s+2)}}$$

A.9 **(10%)** The performance requirements of a chemical process require that its output have a two-percent settling time less than 1 second and an overshoot less than 5 percent. Sketch the region on the real-imaginary plane that satisfies these performance requirements.

$$\%OS = 100\,e^{-\pi\zeta/\sqrt{1-\zeta^2}} < 5\%$$

$$t_s\big|_{2\%} = \frac{4}{\zeta\omega_n} < 1 \Rightarrow \zeta\omega_n > 4$$

$$\frac{-\pi\zeta}{\sqrt{1-\zeta^2}} = \ln 0.05$$

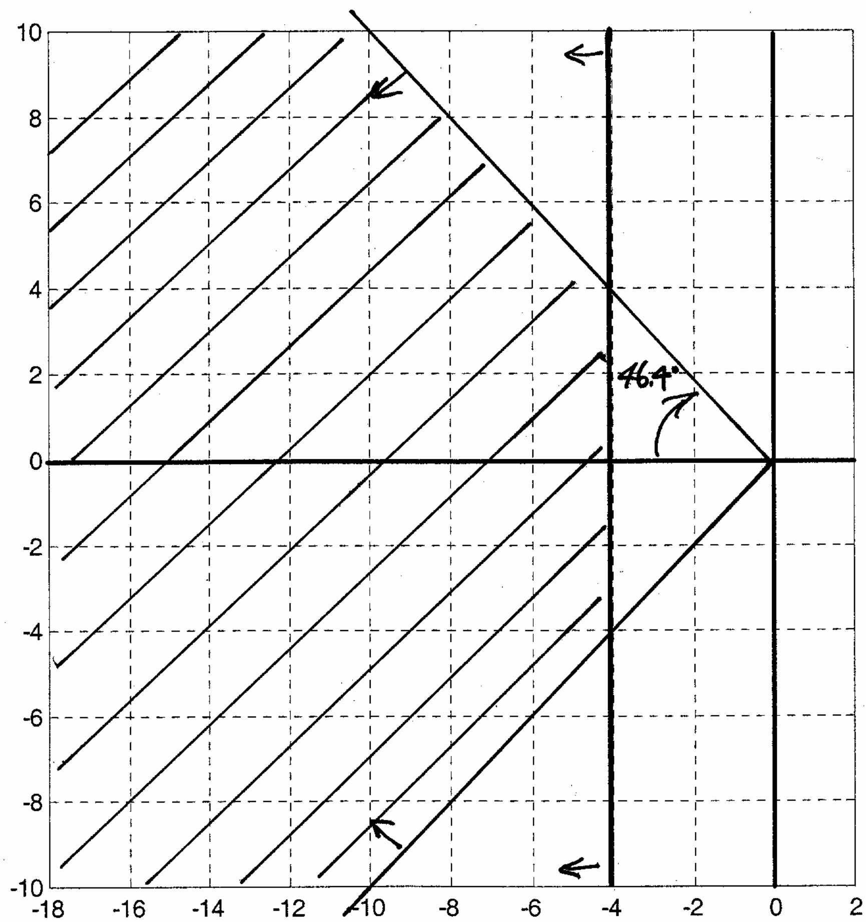
$$\pi^2\zeta^2 = \ln^2 0.05\,(1-\zeta^2)$$

$$(\pi^2 + \ln^2 0.05)\,\zeta^2 = \ln^2 0.05$$

$$\zeta^2 = \frac{\ln^2 0.05}{\pi^2 + \ln^2 0.05}$$

$$\zeta = 0.69$$

$$\theta = \cos^{-1} 0.69 = 46.4^\circ$$



A.10 **(8%)** Find the steady-state response $y_{ss}(t)$ of a system with transfer function $G(s) = \frac{Y(s)}{U(s)} = \frac{36}{s^2+3s+36}$ when the input is $u(t) = 2\sin(6t)$. Make sure you compute all required terms.

$$G(j\omega) = \frac{36}{36-\omega^2+j3\omega}$$

$$y_{ss}(t) = 2\,|G(j6)|\sin(6t + \angle G(j6))$$

$$|G(j\omega)| = \frac{36}{\sqrt{(36-\omega^2)^2+(3\omega)^2}} \Rightarrow |G(j6)| = \frac{36}{18} = 2$$

$$\angle G(j\omega) = -\tan^{-1}\frac{3\omega}{36-\omega^2} \Rightarrow \angle G(j6) = -90^\circ$$

$$\boxed{y_{ss}(t) = 4\sin(6t - 90^\circ)}$$

A.11 **(6%)** Given the following two transfer functions:

(1) $\frac{13}{s^2+6s+13}$ (2) $\frac{46}{s^2+15s+50}$

(a) Determine damping ratio for each system. Which system has more overshoot for a step response?

$\omega_1 = \sqrt{13} = 3.6$ r/s, $\omega_2 = \sqrt{50} = 7.07$ r/s

$2\zeta_1\omega_1 = 6 \Rightarrow \boxed{\zeta_1 = 0.83}$, $2\zeta_2\omega_2 = 15 \Rightarrow \boxed{\zeta_2 = 1.06}$

System (1) has more overshoot.

(b) Determine static gain for each system. Which system has the higher gain?

$G_1(0) = 1$, $G_2(0) = \frac{46}{50} = 0.92$

System (1) has higher gain.

A.12 **(14%)** The cooling system in your automobile engine serves to remove excess heat from the engine block. We will model this system as follows. The combustion process is assumed to provide a heat flow rate $Q_i(t)$ to the engine block, which has total mass m, specific heat c_p, and temperature T_b. Coolant flows through the block and removes heat via convection, with convection coefficient h_i and total interface area A_i. Coolant at temperature T_c flows to the radiator, where it exchanges heat with ambient air at temperature T_a via convection, with convection coefficient h_o and total interface area A_o. Neglect thermal capacitance of the coolant.

Derive a differential equation that describes how the engine block temperature T_b reacts to the heat flow rate $Q_i(t)$.

for convection, $R = \dfrac{1}{hA}$; $C_b = mc_p$

Energy balance:

block: $$C_b \frac{dT_b}{dt} = Q_i - \frac{1}{R_i}(T_b - T_c)$$

coolant: $$\cancel{C_c}^{\,0} \frac{dT_c}{dt} = \frac{1}{R_i}(T_b - T_c) - \frac{1}{R_o}(T_c - T_a)$$

$$\left(\frac{1}{R_i} + \frac{1}{R_o}\right) T_c = \frac{1}{R_i} T_b + \frac{1}{R_o} T_a$$

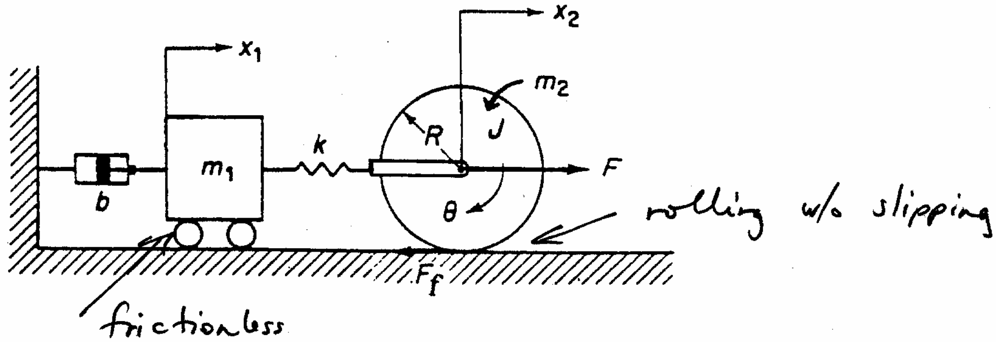
$$\left(\frac{1}{R_i} + \frac{1}{R_o}\right) T_b - \left(\frac{1}{R_i} + \frac{1}{R_o}\right) T_c = \frac{1}{R_o} T_b - \frac{1}{R_o} T_a$$

$$\Rightarrow T_b - T_c = \frac{\frac{1}{R_o}}{\frac{1}{R_i} + \frac{1}{R_o}}(T_b - T_a) = \frac{R_i}{R_i + R_o}(T_b - T_a)$$

$$C_b \frac{dT_b}{dt} = Q_i - \frac{1}{R_i + R_o}(T_b - T_a)$$

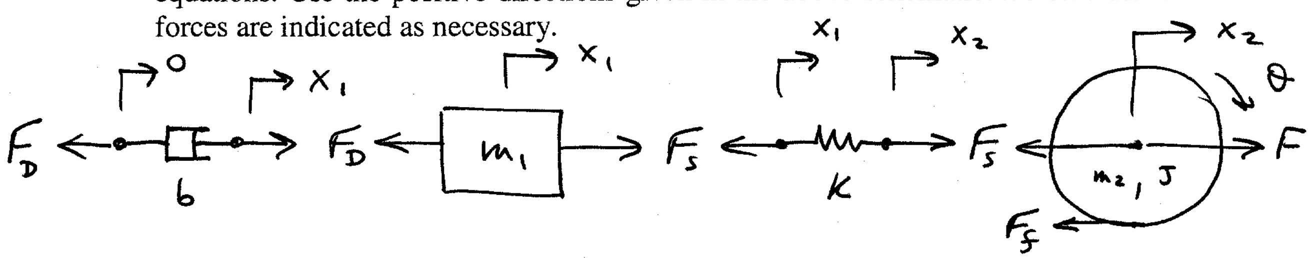
$$\boxed{\underbrace{(R_i + R_o)}_{R_{eq}} C_b \frac{dT_b}{dt} + T_b = (R_i + R_o) Q_i + T_a}$$

B.1 **(20%)** The following schematic shows a rack and pinion system:



In the above schematic, F is the force acting at the center of the disk, which has moment of inertia $J = \frac{1}{2} m_2 R^2$ and mass m_2. The disk has radius R and makes an angle θ with the vertical. Note that $\theta = 0$ when $x_2 = 0$.

B.1(a) **(10%)** Draw free body diagrams of all the elements and write down the elemental equations. Use the positive directions given in the above schematic. Be sure that the motions and forces are indicated as necessary.



$$F_D = b\dot{x}_1 \quad / \quad m_1\ddot{x}_1 = F_S - F_D \quad / \quad F_S = k(x_2 - x_1)$$

$$m_2\ddot{x}_2 = F - F_S - F_f$$

$$J\ddot{\theta} = F_f R \Rightarrow F_f = \frac{J\ddot{\theta}}{R}$$

rolling w/o slipping : $x_2 = R\theta$

(1) $m_1\ddot{x}_1 + b\dot{x}_1 + k(x_1 - x_2) = 0$

$$m_2\ddot{x}_2 + \frac{J\ddot{\theta}}{R} + k(x_2 - x_1) = F$$

(2) $\frac{3}{2} m_2 \ddot{x}_2 + k(x_2 - x_1) = F$

B.1(b) (10%) With F being the input and x_1 being the output, derive the input-output model. You may use the Laplace transform in the process of derivation. However, be sure that the final result is a differential equation.

From (1): $x_2 = \frac{1}{k}\left(m_1 \ddot{x}_1 + b\dot{x}_1 + k_1 x_1\right)$

Substitute into (2):

$$\frac{3}{2} m_2 \left[\frac{1}{k}\left(m_1 \ddddot{x}_1 + b\dddot{x}_1 + k_1 \ddot{x}_1\right)\right]$$

$$+ m_1 \ddot{x}_1 + b\dot{x}_1 + \cancel{k x_1} - \cancel{k x_1} = F$$

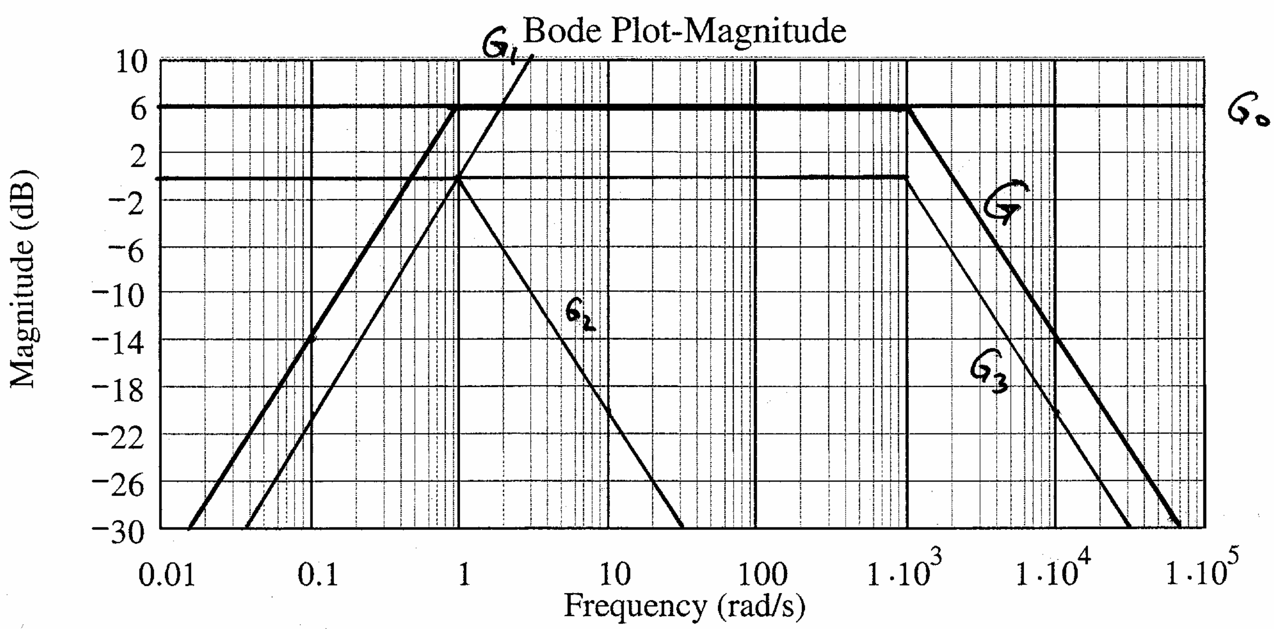
$$\boxed{\frac{3}{2} m_2 m_1 \ddddot{x}_1 + \frac{3}{2} m_2 b \dddot{x}_1 + \left(\frac{3}{2} m_2 k + m_1 k\right)\ddot{x}_1 + bk\dot{x}_1 = F}$$

B.2 **(20%)** Draw a straight-line approximation bode magnitude plot for the system with transfer function: $G(s) = \dfrac{2000s}{s^2+1001s+1000}$. Clearly show the break frequencies.

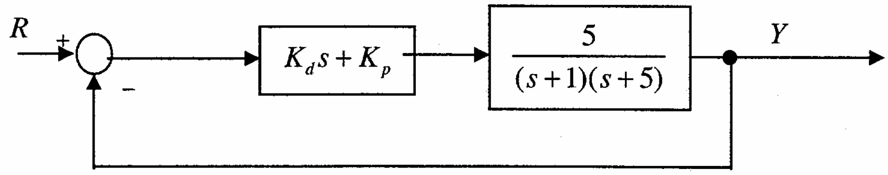
$$G(s) = \frac{2s}{(s+1)(s/1000+1)}$$

$$G_0(s) = 2, \quad G_1(s) = s$$

$$G_2(s) = \frac{1}{s+1}, \quad G_3(s) = \frac{1}{s/1000+1}$$



B.3 **(30%)** Given the following system:



(a) Design PD controller gains K_d and K_p such that the closed-loop system has poles at $s = -7 \pm j7$.

$$G_{cl}(s) = \frac{Y(s)}{R(s)} = \frac{5K_d s + 5K_p}{\underbrace{(s+1)(s+5) + 5K_d s + 5K_p}}$$

$$s^2 + (6+5K_d)s + (5+5K_p) = 0$$

desired char. eq:

$$(s+7+j7)(s+7-j7) = s^2 + 14s + 98$$

equate coefficients:

$$6 + 5K_d = 14 \Rightarrow \boxed{K_d = \frac{8}{5} = 1.6}$$

$$5 + 5K_p = 98 \Rightarrow \boxed{K_p = 18.6}$$

(b) What is the steady-state error for a unit step reference?

$$G_{cl}(0) = \frac{5K_p}{5+5K_p} = y_{ss} \Rightarrow e_{ss} = 1 - \frac{5K_p}{5+5K_p} = \frac{5}{5+5K_p}$$

$$\boxed{e_{ss} = \frac{1}{1+K_p}}$$

(c) If the error is nonzero, suggest a different controller that will eliminate the error.

PI or PID

B.4 (30%) Sketch the root locus for the following characteristic equation:

$$1 + K\frac{s+6}{(s+2)(s-2)} = 0$$

Be sure to calculate (and clearly label) any asymptotes, break-in/breakaway points, and imaginary axis crossings (if any). 1 asymptote → $-180°$

$$K = \frac{-s^2+4}{s+6}$$

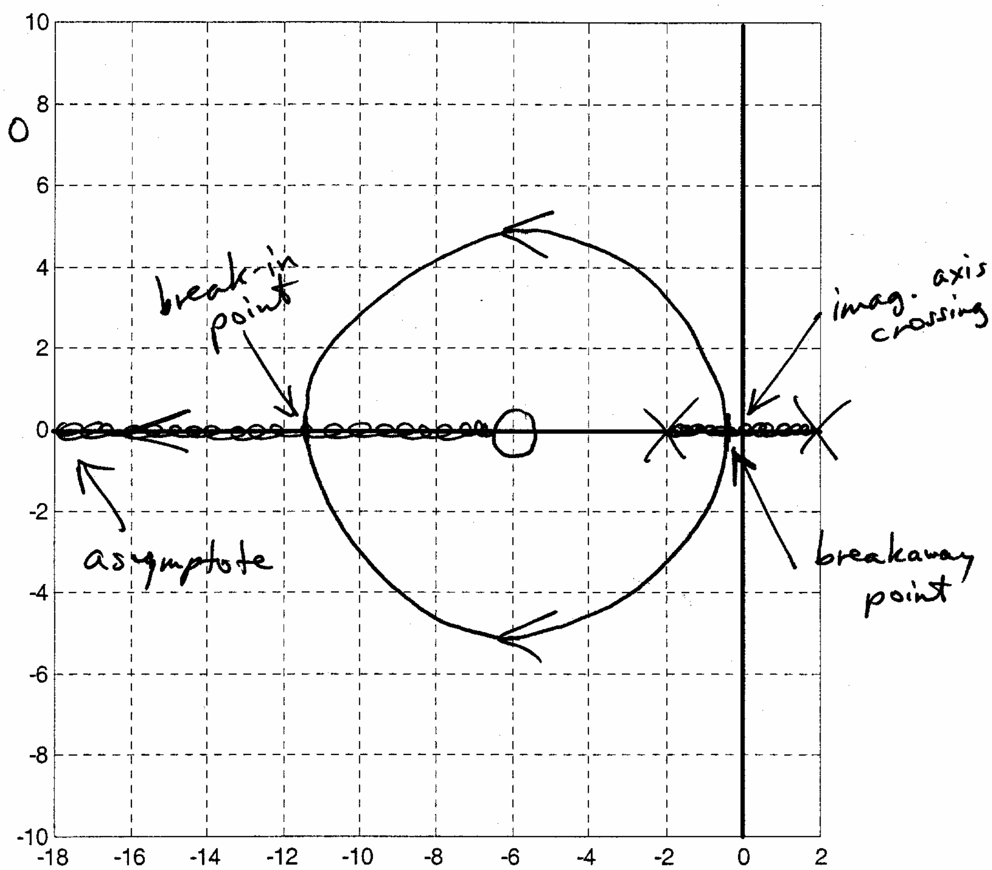
$$\frac{dK}{ds} = \frac{(s+6)(-2s)-(-s^2+4)}{(s+6)^2} = 0$$

$$-2s^2 - 12s + s^2 - 4 = 0$$

$$s^2 + 12s + 4 = 0$$

$$\Rightarrow s = -6 \pm \tfrac{1}{2}\sqrt{144-16}$$

$$s = \begin{cases} -11.66 \\ -0.343 \end{cases}$$



Provide the range of gain K for which this system is stable.

$$s^2 + K(s+6) - 4 = 0$$

Find K where $s = 0$:

$$6K = 4 \Rightarrow K = \frac{2}{3}$$

stable for $\boxed{K > \frac{2}{3}}$