ME 375 FINAL EXAM
Friday, May 6, 2005

Division: King 11:30 / Cunningham 2:30 (circle one) **Name:** SOLUTION

Instructions

(1) This is a closed book examination, but you are allowed three 8.5×11 crib sheets.

(2) You have two hours to work all the problems on the exam.

(3) Use the solution procedure we have discussed: what are you given, what are you asked to find, what are your assumptions, what is your solution, does your solution make sense. You must show all of your work to receive <u>any</u> credit.

(4) You must write neatly and should use a logical format to solve the problems. You are encouraged to really "think" about the problems before you start to solve them. Please write your name in the top right-hand corner of each page.

(5) A table of Laplace transform pairs and properties of Laplace transforms is attached at the end of this exam set.

Section A		Section B	
Problems	Score	Problems	Score
1. (10)		1. (25)	
2. (6)		2. (20)	
3. (6)		3. (30)	
4. (6)		4. (25)	
5. (8)			
6. (15)			
7. (15)			
8. (6)			
9. (6)			
10. (8)			
11. (6)			
12. (8)			
Total (A)	/100	Total (B)	/100
		Total (A+B)	/200

A.1 (10%) Using Laplace transforms, find the total response $y(t)$ of a system with the equation of motion given by:

$$\ddot{y} + 8\dot{y} + 16y = 32u$$

when the input $u(t)$ is a unit step, $y(0) = 0$, and $\dot{y}(0) = 0$.

$$\left[s^2 Y(s) - s\cancel{y(0)}^{0} - \cancel{\dot{y}(0)}^{0}\right] + 8\left[sY(s) - \cancel{y(0)}^{0}\right] + 16\,Y(s) = 32\,U(s)$$

$$\left[s^2 + 8s + 16\right] Y(s) = 32\,U(s)$$

$$Y(s) = \frac{32}{s^2+8s+16}\,U(s) = \frac{32}{s^2+8s+16}\left(\frac{1}{s}\right) = \frac{32}{(s+4)^2}\left(\frac{1}{s}\right)$$

$$Y(s) = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{(s+4)^2}$$

$$= \frac{32}{s(s+4)^2} = \frac{A(s^2+8s+16) + Bs(s+4) + Cs}{s(s+4)^2}$$

s^2: $A + B = 0 \quad \Rightarrow B = -A = -2$

s^1: $8A + 4B + C = 0 \quad \Rightarrow C = -8A - 4B = -16 + 8 = -8$

s^0: $16A = 32 \quad \Rightarrow A = 2$

$\mathcal{L}^{-1}$

$$\boxed{y(t) = \left[2 - 2e^{-4t} - 8te^{-4t}\right]u(t)}$$

A.2 **(6%)** A two-degree-of-freedom mechanical system is described by the following two differential equations:

$$\ddot{x}_1 + 3\dot{x}_1 - 2\dot{x}_2 + 5x_1 - 5x_2 = 0$$
$$4\ddot{x}_2 - 2\dot{x}_1 + 3\dot{x}_2 - 5x_1 + 5x_2 = F$$

Write the equations in Laplace domain and explain how to determine the transfer function $X_1(s)/F(s)$

FOR TRANSFER FUNCTIONS $x_1(0)=0$ $\dot{x}_1(0)=0$
$x_2(0)=0$ $\dot{x}_2(0)=0$

$$\boxed{\begin{aligned}(s^2+3s+5)\,X_1(s) - (2s+5)\,X_2(s) &= 0\\ -(2s+5)\,X_1(s) + (4s^2+3s+5) &= F\end{aligned}}$$

TO GET $\frac{X_1(s)}{F(s)}$, ELIMINATE $X_2(s)$ FROM THE ABOVE EQUATIONS BY COMBINING THEM IN THE LAPLACE DOMAIN. THEN REARRANGE THE RESULTING EQUATION TO GET $\frac{X_1(s)}{F(s)}$. YOU MAY USE CRAMER'S RULE OR ALGEBRA TO COMBINE THE EQUATIONS.

A.3 **(6%)** The motion of a system is governed by $\ddot{y} + 4\dot{y} + 9y = 7u$, where y is the output of the system and u is the input of the system. Give the transfer function from the input to the output. Next, find the poles of the transfer function and determine the stability of this system, giving a **brief** explanation.

$\left.\begin{aligned}\dot{y}(0)&=0\\ y(0)&=0\end{aligned}\right\}$ FOR TRANSFER FUNCTIONS

$$\boxed{\frac{Y(s)}{U(s)} = \frac{7}{s^2+4s+9}}$$

$$s^2+4s+9 = (s^2+4s+4)+5 = (s+2)^2+5 = (s+2)^2+(\sqrt{5})^2$$

$$\boxed{\text{poles: } -2 \pm \sqrt{5}}$$

THIS IS A STABLE SYSTEM BECAUSE THE POLES HAVE NEGATIVE REAL PARTS. (i.e., THE POLES ARE IN THE LEFT-HALF-PLANE.)

A.4 **(6%)** A second order system is described by $\ddot{y} + 2\dot{y} + 25y = 20u(t)$, where y is the output of the system, and u is the input of the system. Find the natural frequency, damping ratio, and static gain of this system.

$$\frac{Y(s)}{U(s)} = \frac{20}{s^2+2s+20} = \frac{K\,\omega_n^2}{s^2+2\zeta\omega_n \cdot s+\omega_n^2}$$

$$\omega_n^2 = 25 \Rightarrow \boxed{\omega_n = 5}$$

$$2\zeta\omega_n = 2 \Rightarrow \zeta = \frac{\cancel{2}}{\cancel{2}\omega_n} = \frac{1}{5} = 0.2 \Rightarrow \boxed{\zeta = 0.2}$$

$$K\omega_n^2 = 20 \Rightarrow K = \frac{20}{25} = \frac{4}{5} = 0.8 \Rightarrow \boxed{K = 0.8}$$

A.5 **(8%)** The steady-state torque-speed relationship for a dc motor is given by the following formula:

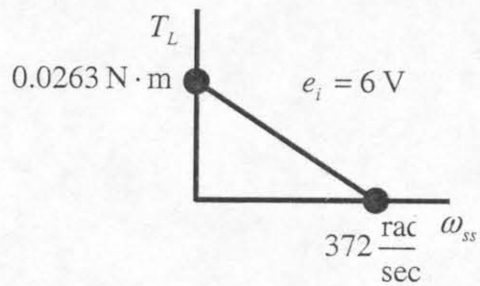
$$T_L = \frac{K_T}{R_A} \cdot e_i - \frac{R_A B + K_T K_b}{R_A} \cdot \omega_{ss}$$

where e_i is the nominal input voltage. Given the torque constant, K_T, the back emf constant, K_b, and the speed load map below, calculate the armature resistance, R_A, and the bearing damping, B.

$\boxed{\text{WITH UNITS!}}$

$$K_T = 0.0144 \frac{\text{N} \cdot \text{m}}{\text{amp}}$$

$$K_b = 0.0144 \frac{\text{V} \cdot sec}{\text{rad}}$$



FOR STALL TORQUE, $\omega_{ss} = 0$

$$T_s = \frac{K_T}{R_A} e_i$$

$$\Rightarrow R_A = \frac{K_T}{T_s} e_i = \frac{0.0144 \frac{\cancel{N \cdot m}}{amp}}{0.0263\, \cancel{N \cdot m}} (6V) = 3.29\,\Omega$$

$$\boxed{R_A = 3.29\,\Omega}$$

FOR NO-LOAD SPEED, $T_L = 0$

$$\frac{K_T}{\cancel{R_A}} e_i - \left(\frac{R_A B + K_T K_b}{\cancel{R_A}}\right) \omega_{ss} = 0$$

$$\frac{V}{amp} = \Omega$$

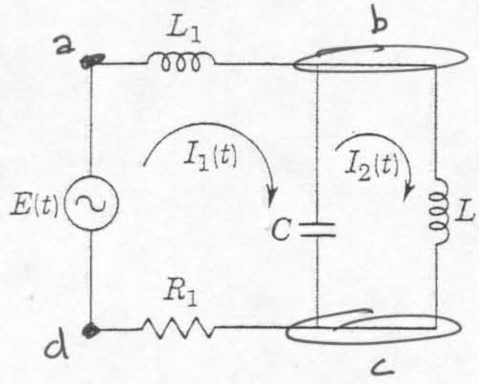
$$R_A B \omega_{ss} = K_T e_i - K_T K_b \omega_{ss}$$

$$B = \frac{1}{R_A \omega_{ss}} \left(K_T e_i - K_T K_b \omega_{ss} \right)$$

$$\Rightarrow \boxed{B = 7.57 \times 10^{-6}\ N \cdot m \cdot \frac{sec}{rad}}$$

$$B = \frac{sec.}{(3.29\,\cancel{\Omega})(372\,rad)} \left[0.0144 \frac{N \cdot m}{\cancel{amp}}\, 6\cancel{V} - (0.0144)^2 \frac{N \cdot m}{\cancel{amp}} \frac{\cancel{V} \cdot \cancel{sec}}{\cancel{rad}} \left(372 \frac{\cancel{rad}}{\cancel{sec}}\right) \right]$$

A.6 **(15%)** Draw a diagram of an analogous mass-spring-damper mechanical system for the electrical system shown in the figure below. Assume force is the effort variable and velocity is the flow variable.

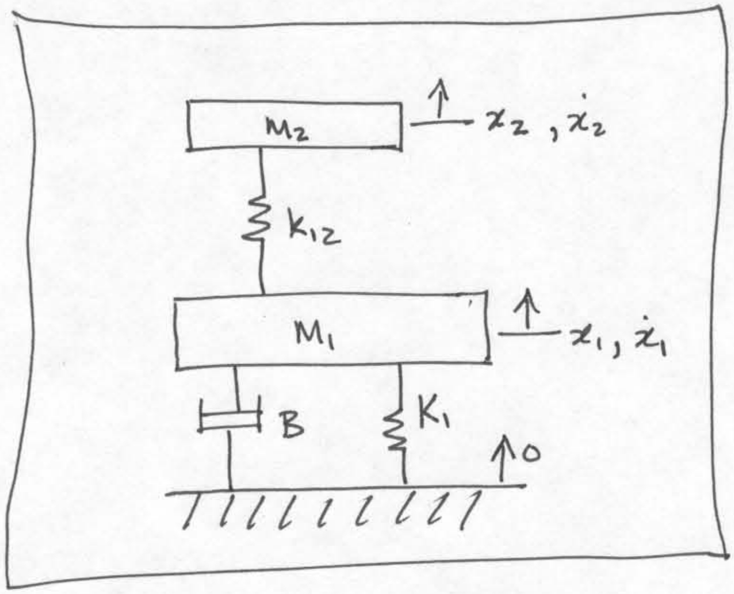


$L_1 \frac{di_1}{dt} = e_{ab}$ $\qquad L_2 \frac{di_2}{dt} = e_{bc}$

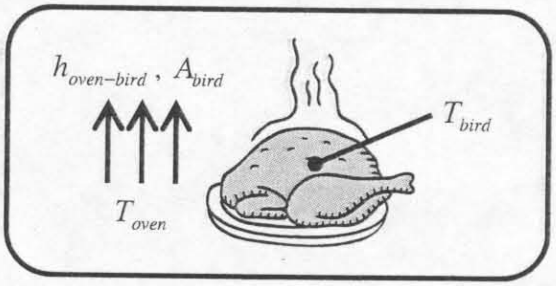
$e_{cd} = R_1 i_1$ $\qquad e_{bc} = \frac{1}{C}\int (i_1 - i_2)\,dt$

$E(V) \Leftrightarrow F\ (N)$

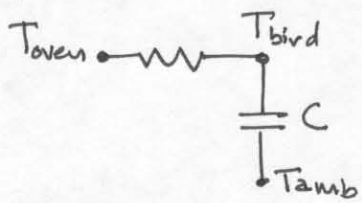
$i\ (\text{amp}) \Leftrightarrow V\left(\frac{m}{s}\right) = \dot{x}$



A.7 **(15%)** As an engineer for an appliance manufacturer, you need to model a turkey roasting in a convection oven. The convective air flow in the oven is kept at the set temperature, T_{oven}, the convective heat transfer coefficient is $h_{oven-bird}$, and the turkey has a heat transfer area, A_{bird}. Assume that as the turkey cooks its temperature is uniform throughout, T_{bird}.



(a) Derive the differential equation that describes the evolution of the temperature of the roasting turkey, T_{bird}, where the input is T_{oven}. (Assume the turkey has heat capacity c_p and mass m_{bird}.)



$$R = \frac{1}{h_{oven\text{-}bird} \cdot A_{bird}}$$

$$C = c_p \cdot m_{bird}$$

ELEMENTAL EQUATIONS

$$q_{in} = \frac{1}{R}(T_{oven} - T_{bird})$$

$$q_{store} = C\frac{dT_{bird}}{dt}$$

ENERGY BALANCE

$$q_{store} = \Sigma q_{in} - \cancel{\Sigma q_{out}}^{\,0} + \cancel{q_{gen}}^{\,0}$$

$$C\frac{dT_{bird}}{dt} = \frac{1}{R}(T_{oven} - T_{bird})$$

$$RC\frac{dT_{bird}}{dt} + T_{bird} = T_{oven}$$

$$\boxed{\left(\frac{c_p \cdot m_{bird}}{h_{oven\text{-}bird} A_{bird}}\right)\frac{dT_{bird}}{dt} + T_{bird} = T_{oven}}$$

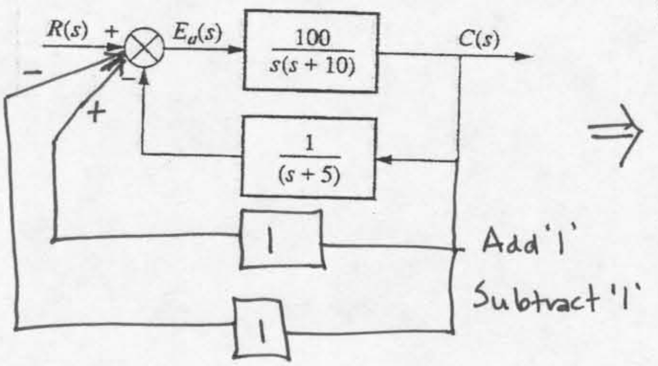
(b) In terms of the model parameters you defined in (a), how long does it take for the turkey temperature, T_{bird}, to rise to within 1% of its final temperature?

$5\tau \rightarrow$ 1% SETTLING TIME

$$\tau = RC = \frac{c_p \cdot m_{bird}}{h_{oven\text{-}bird} \cdot A_{bird}}$$

$$\boxed{5\tau = 5RC = \frac{5\,c_p \cdot m_{bird}}{h_{oven\text{-}bird} \cdot A_{bird}}}$$

A.8 **(6%)** Use block diagram methods to find the corresponding G(s) for a unity feedback system given the non-unity feedback system shown in the figure.



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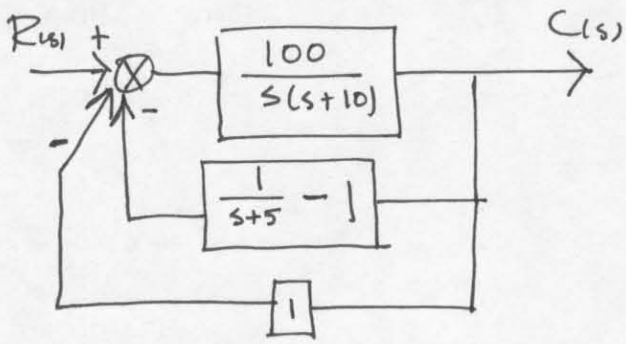

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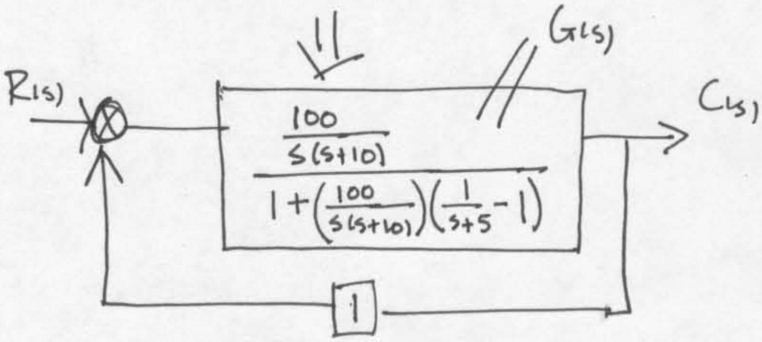


$$G(s) = \frac{\frac{100}{s(s+10)}}{1 + \left(\frac{100}{s(s+10)}\right)\left(\frac{1-s-5}{s+5}\right)}$$

$$\boxed{G(s) = \frac{100(s+5)}{s(s+5)(s+10) - 100(s+4)}}$$

A.9 **(6%)** The performance requirements of an automotive cruise control are less than 2 percent overshoot and lest than 2 seconds two-percent settling time. Sketch the region that satisfies these performance criteria on the complex plane below.

$$t_{s\text{-}2\%} = \frac{4}{\sigma} \leq 2 \text{ sec.}$$

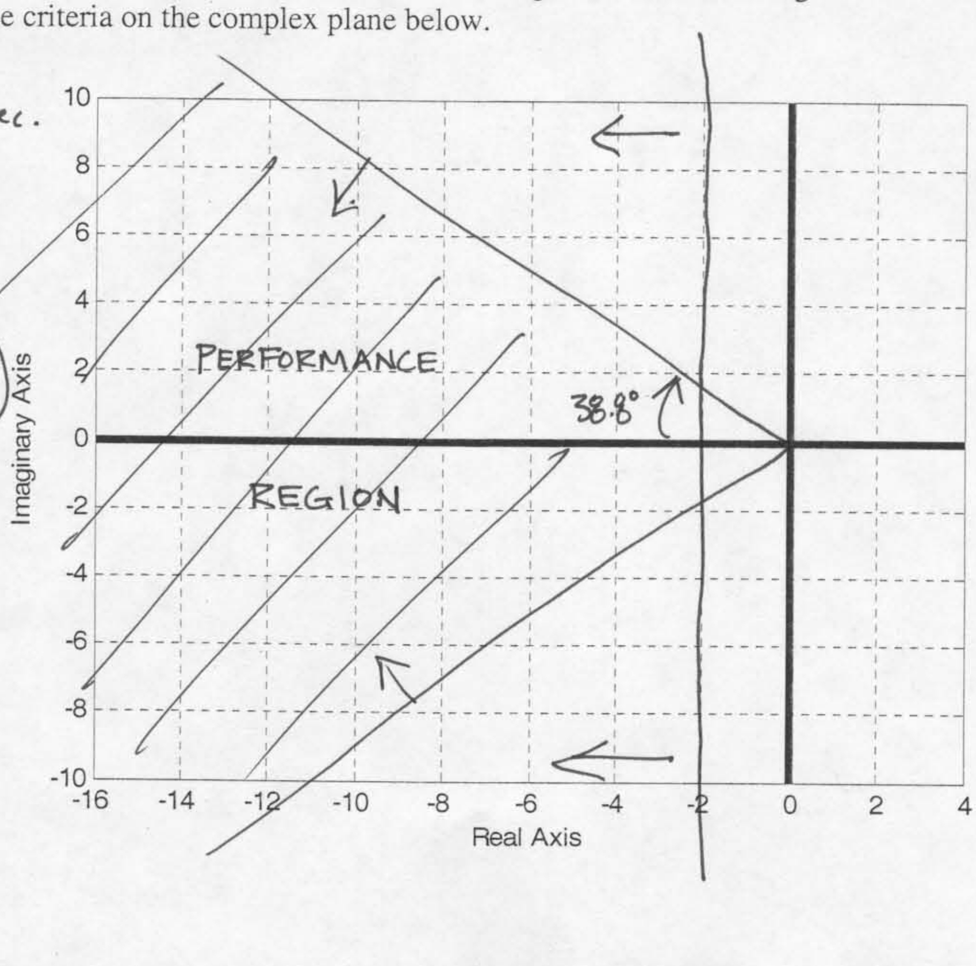
$$\sigma \geq \frac{4}{2}$$

$$\boxed{\sigma \geq 2}$$

$$\theta \leq \tan^{-1}\left(\frac{\pi}{\ln\left(\frac{100}{\%OS}\right)}\right)$$

$$\theta \leq \tan^{-1}\left(\frac{\pi}{\ln(50)}\right)$$

$$\boxed{\theta \leq 38.77^\circ}$$



A.10 **(8%)** Find the steady-state response $y_{ss}(t)$ of a system with transfer function $G(s) = \dfrac{20}{s^2 + 2s + 25}$ when the input is $u(t) = 2\sin(5t)$. Make sure you compute all required terms.

$$G(j\omega) = \frac{20}{-\omega^2 + 2j\omega + 25}$$

$$|G(j\omega)| = \frac{20}{\sqrt{(25-\omega^2)^2 + (2\omega)^2}} \qquad \measuredangle G(j\omega) = -\tan^{-1}\left(\frac{2\omega}{25-\omega^2}\right)$$

$$y_{ss}(t) = |G(j5)|\, 2 \sin\left(5t + \measuredangle G(j5)\right)$$

$$|G(j5)| = \frac{20}{10} = 2 \qquad \measuredangle G(j\omega) = -\tan^{-1}\left(\frac{10}{0}\right) = -90^\circ = -\frac{\pi}{2}$$

$$y_{ss}(t) = (2)(2) \sin\left(5t - \frac{\pi}{2}\right)$$

$$\boxed{y_{ss}(t) = 4 \sin\left(5t - \frac{\pi}{2}\right)}$$

A.11 **(6%)** Given the following transfer function: $G(s) = \dfrac{s+6}{s^2 + 2s + 25}$, determine the ODE if y(t) is the output and u(t) is the input.

$$\frac{Y(s)}{U(s)} = \frac{s+6}{s^2+2s+25} \Rightarrow \left(s^2+2s+25\right)Y(s) = (s+6)\,U(s)$$

$$\xrightarrow{\mathcal{L}^{-1}}$$

$$\boxed{\ddot{y}(t) + 2\dot{y}(t) + 25y(t) = \dot{u}(t) + 6u(t)}$$

A.12 **(8%)** The step response of a standard second-order system (assume $\zeta < 1$) with initial conditions is given by

$$Y(s)=\frac{K\cdot\omega_n^2}{(s+\zeta\omega_n)^2+\omega_d^2}\cdot\left(\frac{1}{s}\right)+\frac{s\cdot y(0)+\dot{y}(0)+2\zeta\omega_n\cdot y(0)}{(s+\zeta\omega_n)^2+\omega_d^2}$$

where $\omega_d=\omega_n\cdot\sqrt{1-\zeta^2}$.

The Partial Fraction Expansion for this system is given below.

$$Y(s)=\overbrace{\underbrace{\frac{A}{s}}_{\text{STEADY-STATE}}+\frac{B\cdot\omega_d}{s^2+2\zeta\omega_n\cdot s+\omega_n^2}+\frac{C\cdot(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\omega_d^2}}^{\text{FORCED}}+\overbrace{\frac{y(0)\cdot(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\omega_d^2}+\frac{\frac{\dot{y}(0)+\zeta\omega_n\cdot y(0)}{\omega_d}\cdot\omega_d}{s^2+2\zeta\omega_n\cdot s+\omega_n^2}}^{\text{FREE (INITIAL CONDITION RESPONSE)}}$$

TRANSIENT

(a) **Above** the Partial Fraction Expansion, identify the **free and forced** portions of the response.

(b) **Below** the Partial Fraction Expansion, identify the **transient and steady-state** portions of the response.

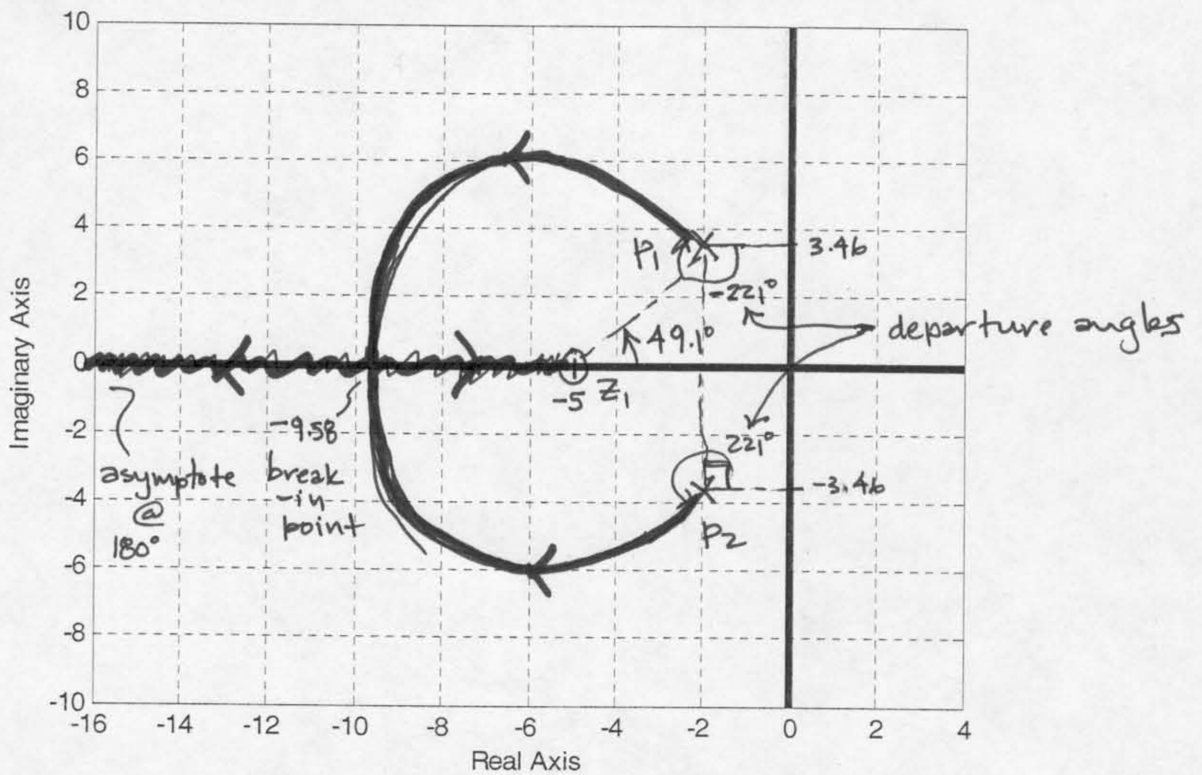
B.1 (25%) Sketch the root locus for the following closed-loop characteristic equation:

$$1+K\cdot\frac{(s+5)}{s^2+4\cdot s+16}=0$$

o.l. zeros: -5

o.l. poles: $-2\pm j\sqrt{12} = -2\pm j\,3.46$

Be sure to calculate (and clearly label) any asympototes, break-in/break-away points, arrival/departure angles, and imaginary axis crossings (if any).



State the range of gain K for which this system is stable.

$\boxed{\text{asymptote}}$ $\theta_k = \frac{(2k+1)\,180°}{2-1} = (2k+1)\,180° \Rightarrow$ one at 180° (neg. real axis)

$\boxed{\text{break-in / break-away points}}$

$$K = -\frac{(s^2+4s+16)}{s+5}$$

$$\frac{dK}{ds} = \frac{-(2s+4)(s+5)+(s^2+4s+16)(1)}{(s+5)^2}=0$$

$$2s^2+14s+20-s^2-4s-16=0$$

$$s^2+10s+4=0 \Longrightarrow s_{1,2}=\begin{cases}\cancel{-0.417}\\ -9.58 \text{ breakin}\end{cases}$$

departure angles

$$\sum_i \angle(s-z_i) - \sum_j \angle(s-p_j) = 180^\circ$$

$$\underbrace{\angle(p_1-z_1)}^{49.1^\circ} - \theta_{p_1} - \overbrace{\angle(p_1-p_2)}^{90^\circ} = 180^\circ$$

$$\angle(p_1-z_1) = \tan^{-1}\left(\frac{3.46}{5-2}\right) = \tan^{-1}\left(\frac{3.46}{3}\right) = 49.1^\circ$$

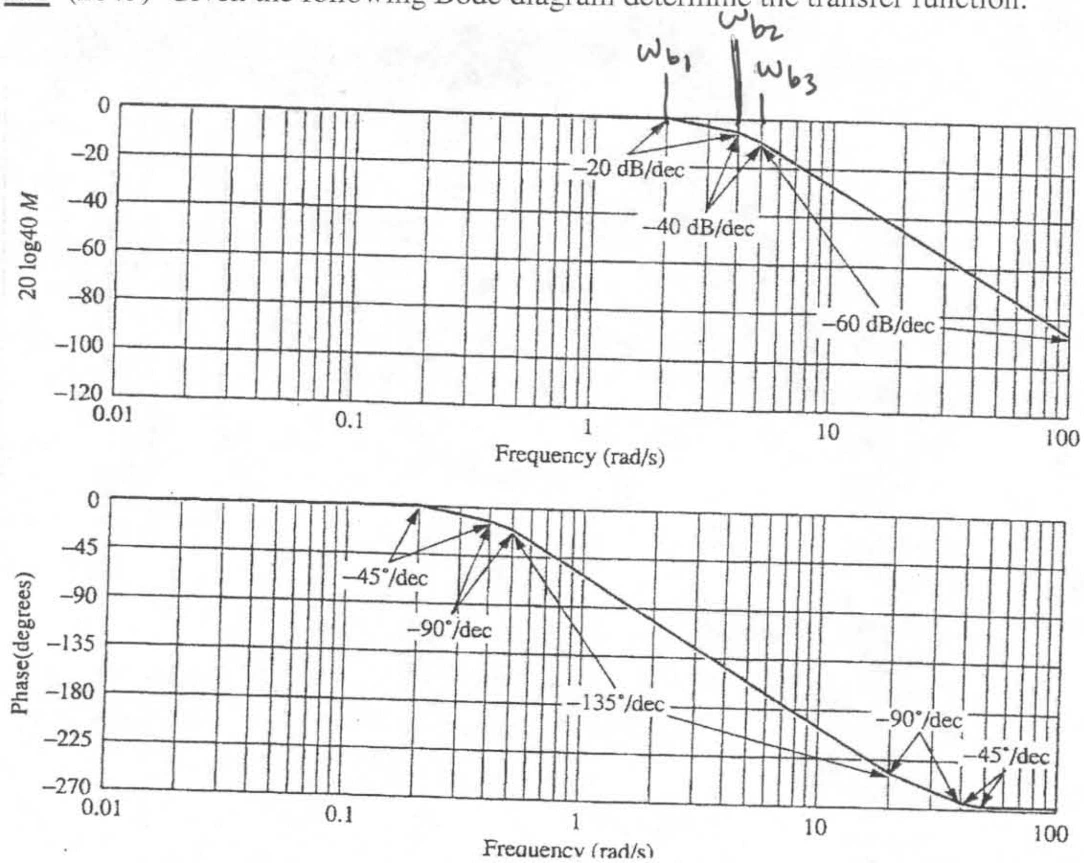
$$\angle(p_1-p_2) = 90^\circ$$

$$\theta_{p_1} = 49.1^\circ - 90^\circ - 180^\circ = -221^\circ$$

$$\boxed{\theta_{p_1} = -221^\circ}$$

No imaginary axis crossings

B.2 **(20%)** Given the following Bode diagram determine the transfer function:



3- FIRST ORDER SYSTEMS (ALL POLES ⇒ $-20 \frac{dB}{dec}$ SLOPES @ HIGH ω)

- STATIC GAIN ⇒ $\boxed{K=1}$ (0 dB @ LOW FREQUENCY)

NO INTEGRATORS/DIFFERENTIATORS

$$\omega_{b1} = \frac{1}{\tau_1} = 2$$

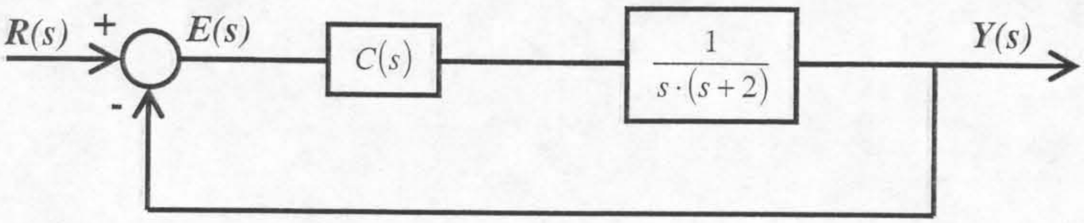
$$\omega_{b2} = \frac{1}{\tau_2} = 4$$

$$\omega_{b3} = \frac{1}{\tau_3} = 5$$

$$G(s) = \frac{1}{(\tau_1 s+1)(\tau_2 s+1)(\tau_3 s+1)}$$

$$\boxed{G(s) = \frac{1}{(\frac{1}{2}s+1)(\frac{1}{4}s+1)(\frac{1}{5}s+1)} = \frac{(2)(4)(5)}{(s+2)(s+4)(s+5)}}$$

B.3 **(30%)** Given the following dc motor feedback system with controller $C(s)$,

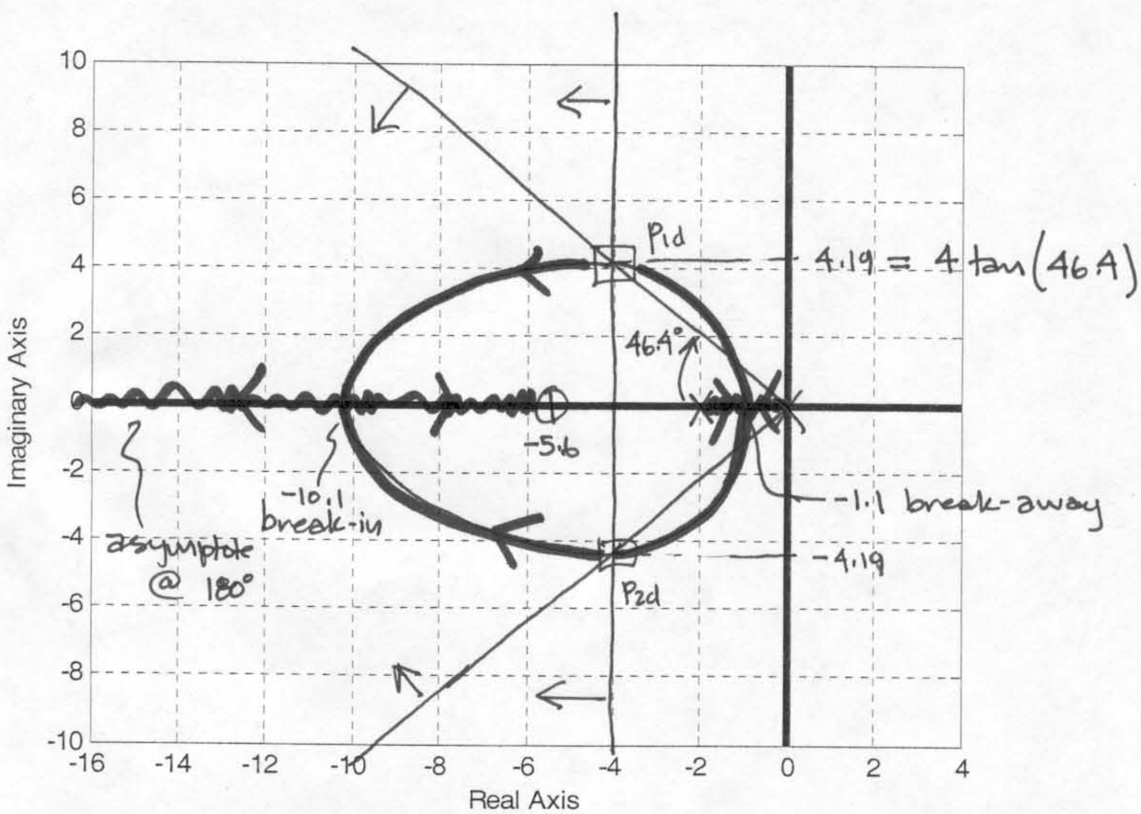


design a PD controller, $C(s) = K_p + K_d \cdot s$, using the following steps.

(a) Determine the desired performance region on the complex plane below using the following performance criteria:

$t_{s-2\%} \leq 1\,\text{sec}$ $\quad t_{s-2\%} = \frac{4}{\sigma} \leq 1\,\text{sec} \Rightarrow \sigma \geq 4$

$\%OS \leq 5\%$ $\quad \theta \leq \tan^{-1}\left(\frac{\pi}{\ln\left(\frac{100}{\%OS}\right)}\right) \Rightarrow \theta \leq 46.4^\circ$



(b) Use the intersection points of the performance constraint lines as the desired closed-loop pole locations, p_{1d} and p_{2d}, and use these to write the **desired** closed-loop characteristic equation. (Hint: $(s - p_{1d})\cdot(s - p_{2d}) = 0$ where p_{1d} and p_{2d} may be complex conjugates.) Multiply the equation out to get a single polynomial.

$$p_{1,2d} = -4 \pm j\,4\tan(46.4^\circ) \Rightarrow \boxed{p_{1,2d} = -4 \pm j\,4.19}$$

$$(s-p_{1d})(s-p_{2d}) = (s+4)^2 + (4.19)^2 = s^2+8s+16+17.6=0$$

$$\boxed{s^2 + 8s + 33.6}$$

(c) Write out the **actual** closed-loop characteristic equation where you let $C(s) = K_p + K_d \cdot s$. Leave K_p and K_d as variables in your polynomial.

$$1 + \frac{K_p + K_d s}{s(s+2)} = 0$$

$$s(s+2) + (K_p + K_d s) = 0 \Rightarrow \boxed{s^2 + (2+K_d)s + K_p = 0}$$

(d) Compare coefficients between the **desired** closed-loop characteristic equation and the **actual** closed-loop characteristic equation to calculate the values for K_p and K_d. This method ensures that the root locus will pass through the desired closed-loop pole locations.

$$2 + K_d = 8 \Rightarrow \boxed{K_d = 6}$$

$$\boxed{K_p = 33.6}$$

$$C(s) = K_p + K_d(s) = K_d\left(s + K_p/K_d\right)$$

$$= 6(s+5.6) = K(s+a) = K_d(s+5.6) \Longrightarrow$$

(f) Finally, using the values for K_p and K_d you calculated in (d) draw the root locus for this system. Remember that $C(s) = K_d \cdot \left(s + {K_p}/{K_d} \right) = K \cdot (s+a)$ such that the controller gain is defined by $K = K_d$ and the controller zero location is defined by $a = {K_p}/{K_d}$.

break-in/break-away points

$$K = \frac{-s(s+2)}{s+5.6} = \frac{-(s^2+2s)}{s+5.6}$$

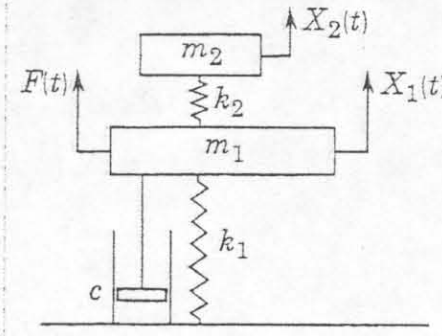
$$\frac{dK}{ds} = -\frac{(2s+2)(s+5.6) + (s^2+2s)(1)}{(s+5.6)^2} = 0$$

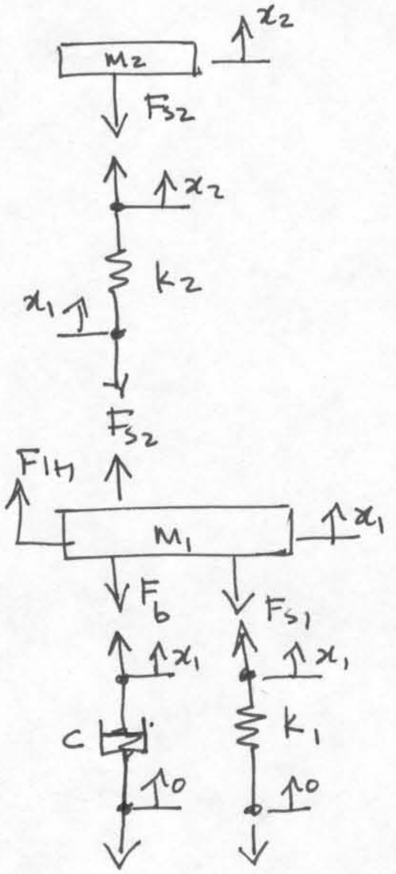
$$2s^2 + 13.2s + 11.2 - s^2 - 2s = 0$$

$$s^2 + 11.2s + 11.2 = 0$$

$$s_{1,2} = \begin{cases} -1.1 & \text{- break-away} \\ -10.1 & \text{- break-in} \end{cases}$$

B.4 **(25%)** Given the following mechanical system: Horizontal Plane





(a) Draw the free body diagrams:

(b) Write the elemental equations

$$F_{s1} = k_2(x_2 - x_1)$$

$$F_{s2} = k_1(x_1 - 0) = k_1 x_1$$

$$F_b = c(\dot{x}_1 - 0) = c\dot{x}_1$$

(c) Determine the equations of motions

$$\sum F_{m_2} = -F_{s2} = m_2\ddot{x}_2$$

$$-k_2(x_2 - x_1) = m_2\ddot{x}_2$$

$$\boxed{m_2\ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0}$$

$$\sum F_{m_1} = F(t) + F_{s2} - F_b - F_{s1} = m_1\ddot{x}_1$$

$$F(t) + k_2(x_2 - x_1) - c\dot{x}_1 - k_1 x_1 = m_1\ddot{x}_1$$

$$\boxed{m_1\ddot{x}_1 + c\dot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = F(t)}$$

Laplace Transform Pairs

$f(t)$	$F(s)$	Comment
$\delta(t)$	1	Unit Impulse
1 for $t > 0$	$\frac{1}{s}$	Unit Step
t for $t > 0$	$\frac{1}{s^2}$	Unit Ramp
e^{-at}	$\frac{1}{s+a}$	Exponential
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	Sine
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	Cosine
$f(t)$	$F(s)$	Function
$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$	First Derivative
$\frac{d^n f(t)}{dt_n}$	$s^n F(s) - s^{n-1} f(0^-) - s^{n-2}\frac{df(0^-)}{dt}\ldots$	n^{th} Derivative
t^n	$\frac{n!}{s^{n+1}}$	
$e^{-at} f(t)$	$F(s+a)$	
te^{-at}	$\frac{1}{(s+a)^2}$	
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$	
$e^{-at}\cos(\omega t)$	$\frac{(s+a)}{(s+a)^2+\omega^2}$	
Initial Value Theorem	$f(0^+) = \lim_{s\to\infty} sF(s)$	
Final Value Theorem	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$	