ME 375 FINAL EXAM
Friday, May 4, 2007

Division Shaver 11:30 / Meckl 2:30 (circle one)   **Name** Solution

Instructions

(1) This is a closed book examination, but you are allowed three 8.5×11 crib sheets.

(2) You have two hours to work all six problems on the exam.

(3) Use the solution procedure we have discussed: what are you given, what are you asked to find, what are your assumptions, what is your solution, does your solution make sense. You must show all of your work to receive any credit.

(4) Circle or box-in your answers.

(5) You must write neatly and should use a logical format to solve the problems. You are encouraged to really "think" about the problems before you start to solve them. Please write your name in the top right-hand corner of each page.

(6) A table of Laplace transform pairs and properties of Laplace transforms is attached at the end of this exam set.

Problem No. 1 (30)	____________
Problem No. 2 (30)	____________
Problem No. 3 (30)	____________
Problem No. 4 (40)	____________
Problem No. 5 (30)	____________
Problem No. 6 (30)	____________
TOTAL (*/200)**	____________

PROBLEM #1: (30 points)

For the following transfer functions:

(i) determine the poles

(ii) compute time constant, natural frequency and/or damping ratio

(iii) indicate which time response (1, 2, 3, or 4) below represents the unit step response of each transfer function.

(a) $\dfrac{25}{s^2+3s+25}$ (2)

poles: $s = -\frac{3}{2} \pm j\frac{\sqrt{91}}{2}$

$\omega_n = 5$ r/s

$2\zeta\omega_n = 3 \Rightarrow \zeta = 0.3$

(b) $\dfrac{10}{s^2+11s+10} = \dfrac{10}{(s+1)(s+10)}$ (4)

poles: $s = -1, -10$

$\tau_1 = 1, \tau_2 = \frac{1}{10}$ sec.

(c) $\dfrac{25}{s^2+6s+25}$ (3)

poles: $s = -\frac{6}{2} \pm j\frac{8}{2}$

$s = -3 \pm j4$

$\omega_n = 5$ r/s

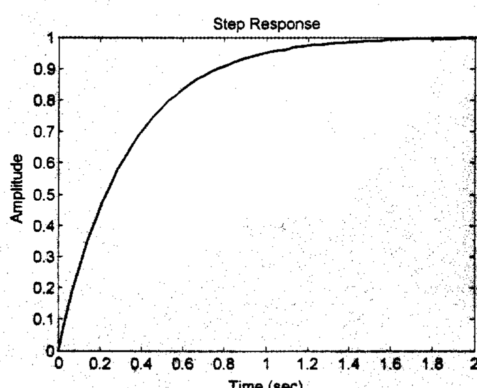
$2\zeta\omega_n = 6 \Rightarrow \zeta = 0.6$

(d) $\dfrac{3}{s+3}$ (1)

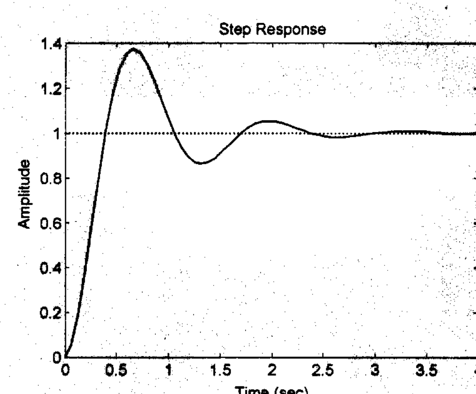
pole: $s = -3$

$\tau = \frac{1}{3}$ sec.

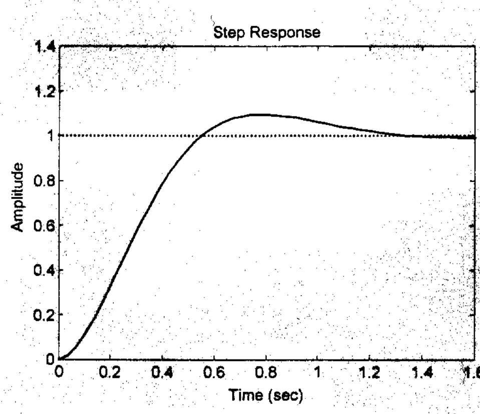
(1)



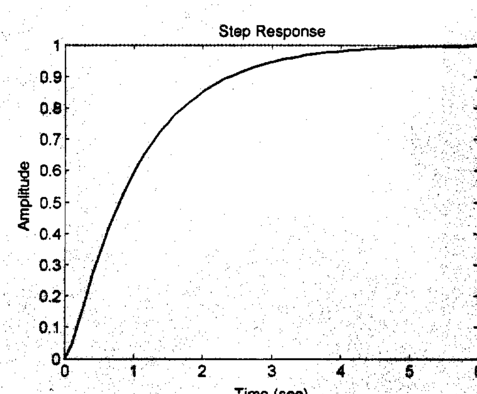
(2)



(3)



(4)



PROBLEM #2: (30 points)

The transfer function for a mechanical system is given by:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{40(s+10)}{(s+1)(s^2+4s+400)}$$

(a) Write down expressions for $G(j\omega)$, $|G(j\omega)|$, and $\angle G(j\omega)$.

$$G(j\omega) = \frac{40\,(j\omega+10)}{(j\omega+1)(400-\omega^2+j4\omega)}$$

$$|G(j\omega)| = \frac{40\sqrt{\omega^2+10^2}}{\sqrt{\omega^2+1^2}\,\sqrt{(400-\omega^2)^2+(4\omega)^2}}$$

$$\angle G(j\omega) = \tan^{-1}\frac{\omega}{10} - \tan^{-1}\omega - \tan^{-1}\frac{4\omega}{400-\omega^2}$$

PROBLEM #2 (cont.)

(b) Find the magnitude and phase of $G(j\omega)$ at $\omega = 10$ and 50 rad/sec.

$\omega = 10$ r/s:

$$|G(j10)| = 0.186$$

$$\angle G(j10) = -46.9^\circ = -0.82 \text{ rad}$$

$\omega = 50$ r/s:

$$|G(j50)| = 0.019$$

$$\angle G(j50) = \tan^{-1} 5 - \tan^{-1} 50 - \underbrace{\tan^{-1}\frac{200}{-2100}}_{180^\circ - \tan^{-1}\frac{200}{2100}}$$

$$\angle G(j50) = -184.7^\circ = -3.224 \text{ rad.}$$

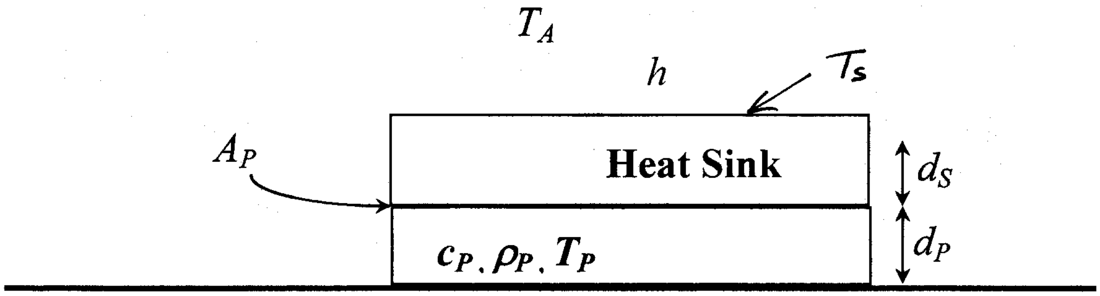
(c) Find the steady-state output $y_{ss}(t)$ of the system to an input of $u(t) = 3 + \sin(10t) + 2\sin(50t)$.

$$u(t) = 3\underbrace{|G(0)|}_{1} + |G(j10)|\sin(10t + \angle G(j10)) + 2|G(j50)|\sin(50t + \angle G(j50))$$

$$\boxed{u(t) = 3 + 0.186\sin(10t - 0.82) + 0.038\sin(50t - 3.224)}$$

PROBLEM #3: (30 points)

A computer chip with added heat sink is represented in the schematic diagram below:



The mass density of the chip material is represented by ρ_P, c_P represents the specific heat of the chip material, T_P is the temperature of the chip, d_P is the height of the chip, and A_P is its top surface area. The heat sink has height d_S. Assume that only convective heat transfer (with convective coefficient h) occurs to the surroundings, which are at ambient temperature T_A. Conductive heat transfer (with thermal conductivity α) occurs between the chip and the heat sink. Ignore any heat transfer through the sides and bottom of the chip and the sides of the heat sink.

The chip is turned on and begins generating heat at a rate given by q. Derive a differential equation that describes the time response of the chip temperature T_P as function of q and T_A.

Conservation of energy: Heat stored = heat supplied − heat lost

$$C\frac{dT_p}{dt} = q - q_{out} \quad , \quad \text{where } C = \rho_p d_p A_p c_p$$

$$q_{out} = \overbrace{\left(\frac{\alpha A_p}{d_s}\right)}^{\frac{1}{R_\alpha}}(T_p - T_s) = \overbrace{(h A_p)}^{\frac{1}{R_h}}(T_s - T_A)$$

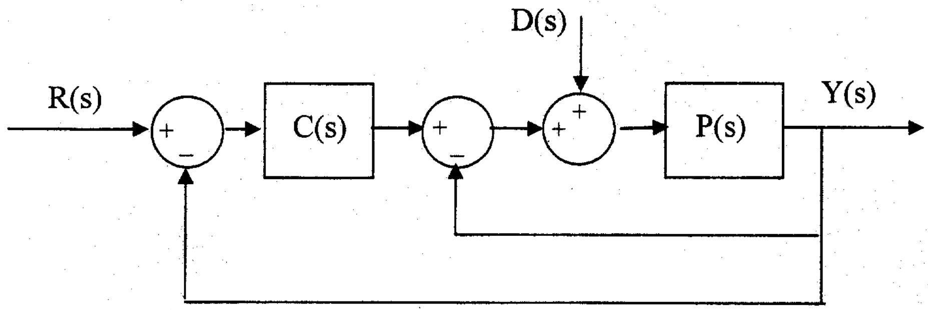
$$T_p - T_s = R_\alpha q_{out}$$
$$T_s - T_A = R_h q_{out}$$
$$T_p - T_A = \underbrace{(R_\alpha + R_h)}_{R_{eq}} q_{out} \Rightarrow q_{out} = \frac{T_p - T_A}{R_{eq}}$$

$$C\frac{dT_p}{dt} = q - \frac{T_p - T_A}{R_{eq}} \Rightarrow R_{eq} C \frac{dT_p}{dt} = R_{eq} q - T_p + T_A$$

$$\boxed{\begin{gathered} R_{eq} C \frac{dT_p}{dt} + T_p = R_{eq} q + T_A \\ \text{where} \quad C = \rho_p d_p A_p c_p \\ R_{eq} = \frac{d_s}{\alpha A_p} + \frac{1}{h A_p} \end{gathered}}$$

PROBLEM #4: **(40 points)**

Consider the following feedback control system:



Where: $C(s) = (s + 10)(s + 40)$ and $P(s) = 100/(s^2 + 3s + 100)$

What is the transfer function:

a) from R to Y, $G_R(s)$ (set $D = 0$)

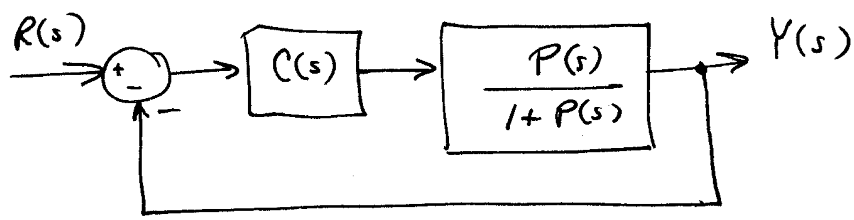
b) from D to Y, $G_D(s)$ (set $R = 0$)

Find the steady-state output y(t) when:

c) r(t) = unit step, d(t) = 0

d) r(t) = 0, d(t) = unit step

(a) with $D = 0$:



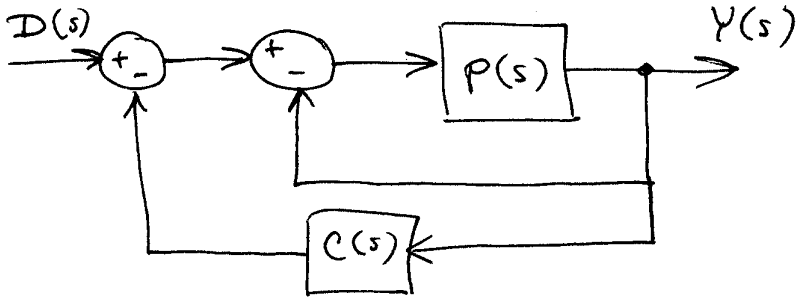
$$G_R(s) = \frac{Y(s)}{R(s)} = \frac{\frac{CP}{1+P}}{1+\frac{CP}{1+P}} = \frac{CP}{1+P+CP}$$

$$G_R(s) = \frac{\frac{(s+10)(s+40)\,100}{s^2+3s+100}}{1+\frac{100}{s^2+3s+100}+\frac{(s+10)(s+40)\,100}{s^2+3s+100}}$$

$$\boxed{\begin{aligned} G_R(s) &= \frac{100\,(s+10)(s+40)}{s^2+3s+100+100+100\,(s+10)(s+40)} \\ G_R(s) &= \frac{100s^2+5000s+40{,}000}{101s^2+5003s+40{,}200} \end{aligned}}$$

PROBLEM #4 (cont.)

(b) with $R=0$, redraw block diagram:



note that the order of the summing junctions was interchanged

$$G_D(s) = \frac{Y(s)}{D(s)} = \frac{\dfrac{P(s)}{1+P(s)}}{1+\dfrac{C(s)P(s)}{1+P(s)}} = \frac{P(s)}{1+P(s)+C(s)P(s)}$$

$$G_D(s) = \frac{\dfrac{100}{s^2+3s+100}}{1+\dfrac{100}{s^2+3s+100}+\dfrac{(s+10)(s+40)/100}{s^2+3s+100}} = \frac{100}{s^2+3s+100+100+100(s+10)(s+40)}$$

$$\boxed{G_D(s) = \frac{100}{101s^2+5003s+40{,}200}}$$

(c) Use Final Value Theorem:

$$y_{ss}(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\,G_R(s)\cancelto{\frac{1}{s}}{R(s)} = \lim_{s\to 0} G_R(s)$$

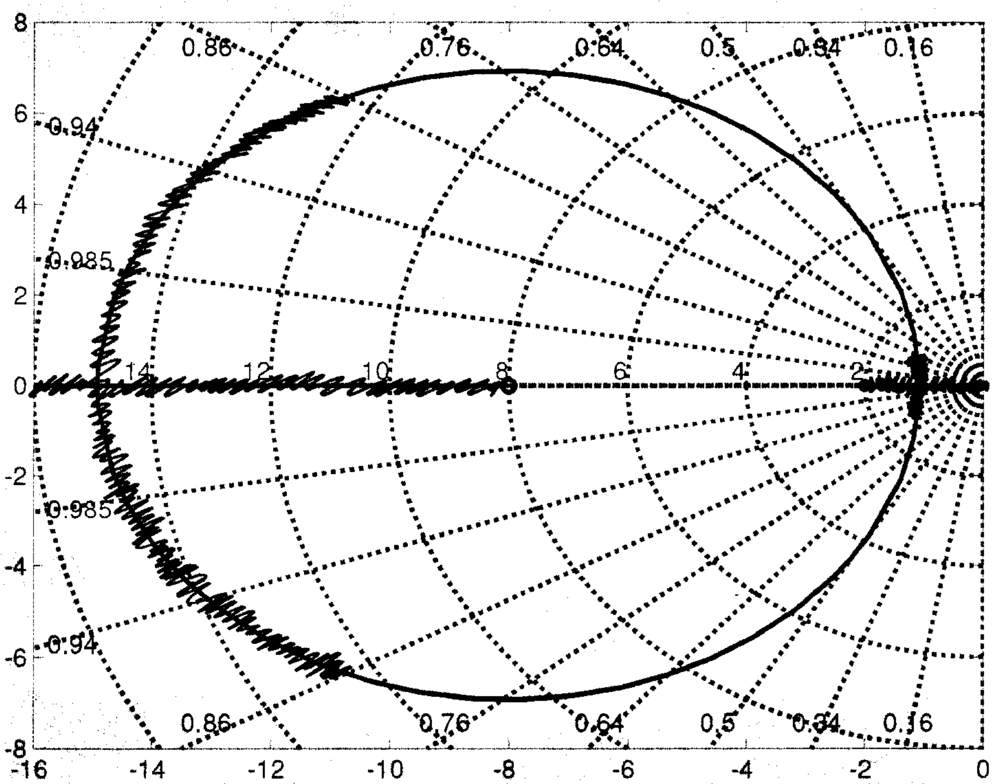
$$\boxed{y_{ss}(t) = \frac{40{,}000}{40{,}200} = \frac{200}{201}}$$

(d)
$$y_{ss}(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\,G_D(s)\cancelto{\frac{1}{s}}{D(s)} = \lim_{s\to 0} G_D(s)$$

$$\boxed{y_{ss}(t) = \frac{100}{40{,}200} = \frac{1}{402}}$$

PROBLEM #5: (30 points)

(a) Indicate the pole locations for the following root locus that correspond to a %OS ≤ 0.5%



$$\%OS = 100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}} \leq 0.5$$

$$e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 0.005$$

$$-\pi\zeta/\sqrt{1-\zeta^2} = \ln 0.005$$

$$\pi^2\zeta^2 = (\ln 0.005)^2(1-\zeta^2)$$

$$(\pi^2+(\ln 0.005)^2)\,\zeta^2 = (\ln 0.005)^2$$

$$\zeta^2 = \frac{(\ln 0.005)^2}{\pi^2+(\ln 0.005)^2} = 0.74$$

$$\zeta = 0.86$$

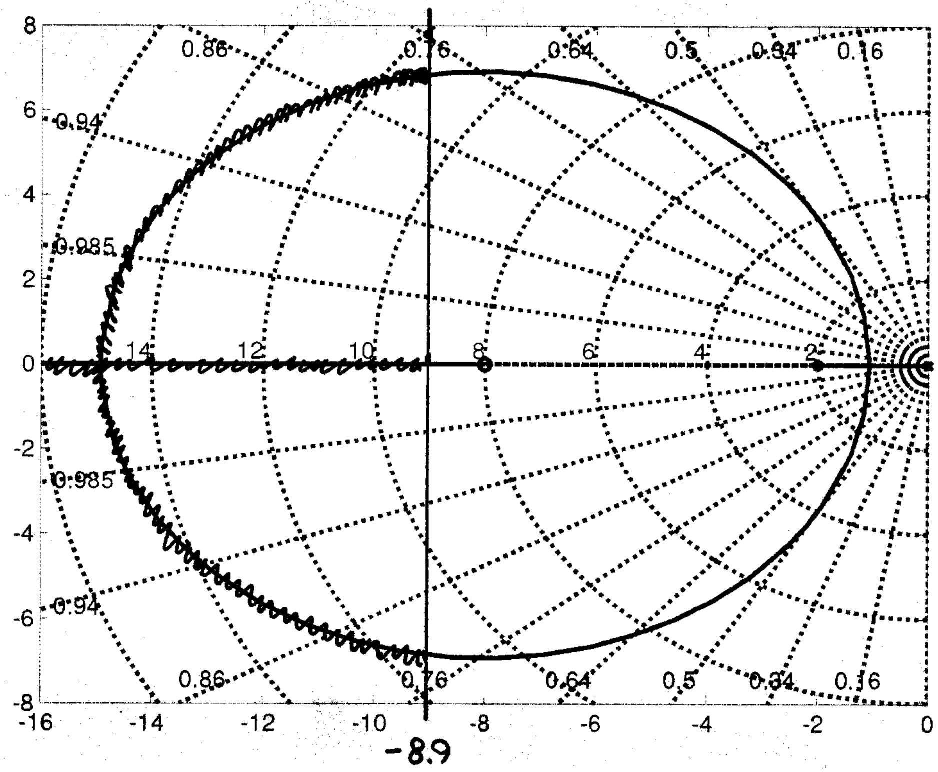
To satisfy requirement, $\zeta \geq 0.86$

(b) Indicate the pole locations for the following root locus that correspond to a 2% settling time ≤ 0.45 seconds.

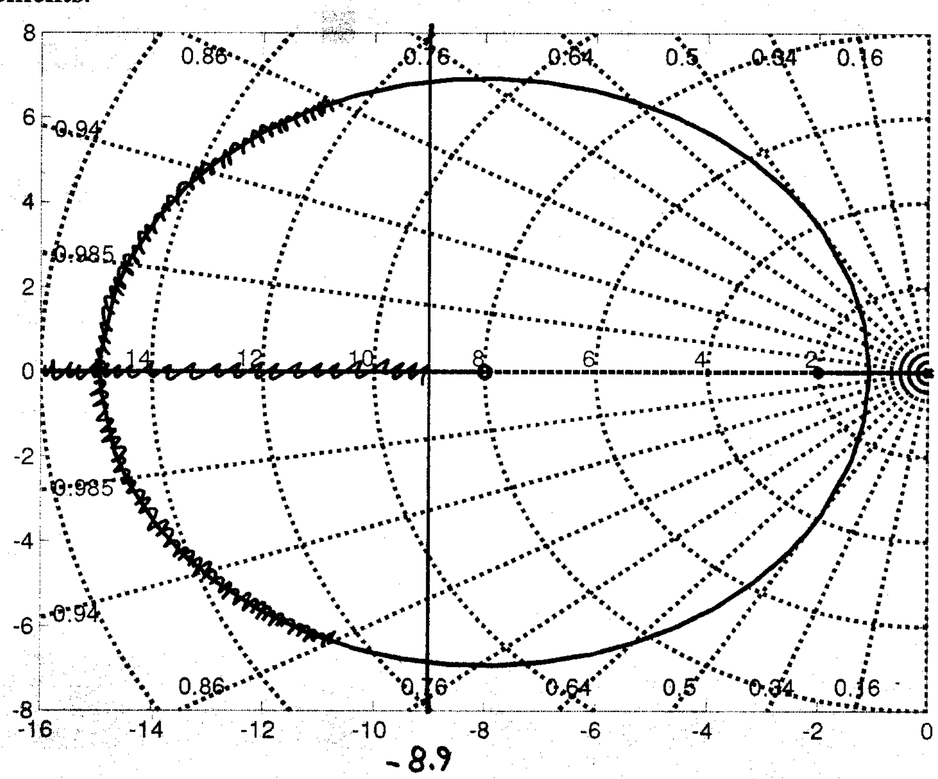
$$\frac{4}{\zeta\omega_n} \leq 0.45$$

$$\zeta\omega_n \geq \frac{4}{0.45}$$

$$\zeta\omega_n \geq 8.9$$

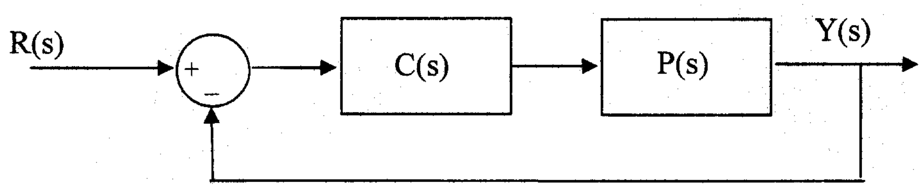


(c) Indicate the pole locations for the following root locus that correspond to both requirements.



PROBLEM #6: **(30 points)**

Consider the following feedback control system:



where $P(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$, with $\omega_n = 2$ rad/s, $\zeta = 0.1$

Design a PD controller C(s) so that the resulting closed-loop system has damping ratio of 0.7 and natural frequency of 4 rad/s.

$$\frac{Y(s)}{R(s)} = \frac{CP}{1+CP} = \frac{\dfrac{(K_p + K_d s)\ 4}{s^2 + 0.4s + 4}}{1 + \dfrac{(K_p + K_d s)\ 4}{s^2 + 0.4s + 4}}$$

$$= \frac{4(K_d s + K_p)}{s^2 + 0.4s + 4 + 4(K_d s + K_p)} = \frac{4(K_d s + K_p)}{s^2 + \underbrace{(0.4 + 4K_d)}_{2\zeta_{cc}\omega_{cc}} s + \underbrace{(4 + 4K_p)}_{\omega_{cc}^2}}$$

where ζ_{cc} and ω_{cc} are closed-loop damping ratio & natural frequency, respectively.

Thus, $0.4 + 4K_d = 2\zeta_{cc}\omega_{cc} = 2(0.7)(4) = 5.6$

$$4K_d = 5.2 \Rightarrow \boxed{K_d = 1.3}$$

$$4 + 4K_p = \omega_{cc}^2 = 16$$

$$4K_p = 12 \Rightarrow \boxed{K_p = 3}$$