Exam 2

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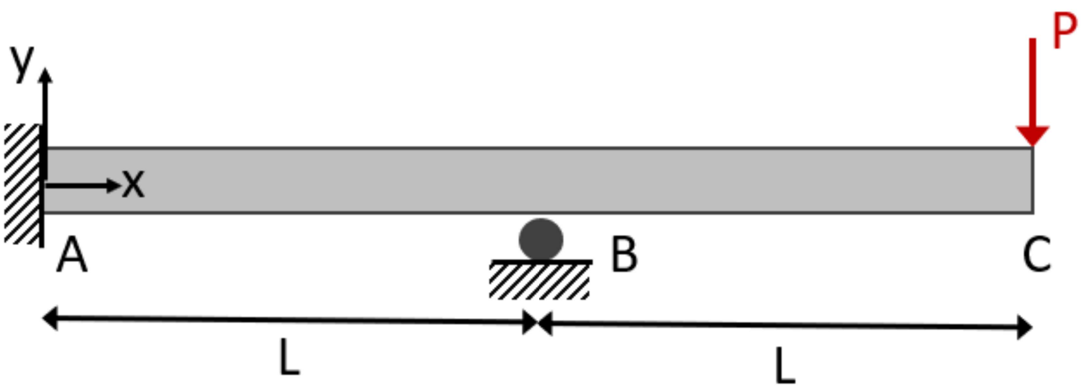
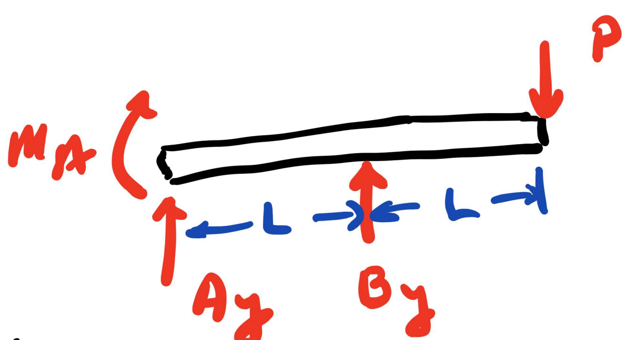
PROBLEM #1 (25 points)

Beam ABC is fixed at end A and is supported by a roller support at B. A concentrated force P acts at C. E and I are constant along the beam. **Use the second order integration method** to calculate the following:

(a) Draw a free body diagram and write the equilibrium equations.
(b) Find the reactions on the beam at A and B in terms of P.
(c) Find the equation for the vertical displacement, v(x) using the x-direction shown in the figure, throughout the beam in terms of P, L, E, and I.
(d) Find the slope (θ) at point B in terms of P, L E, and I.



FBD



Equilibrium:

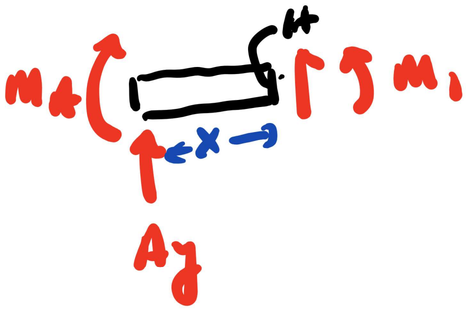
$$\Sigma F_y = A_y + B_y - P = 0 \quad (1)$$

$$\Sigma M_A = -M_A + B_y L - P(2L) = 0 \quad (2)$$

2 equations, 3 unknowns: indeterminate

Deflections

$0 < x < L$



$$\Sigma M_H = M_1 - M_A - A_y x = 0$$

$$M_1(x) = M_A + A_y x = EI \frac{d\theta_1}{dx}$$

0 B.C.

$$\theta_1(x) = \cancel{\theta_1(0)} + \frac{1}{EI}\int_0^x (M_A + A_y x)\,dx$$

$$\theta_1(x) = \frac{1}{EI}\left[M_A x + A_y \frac{x^2}{2}\right] = \frac{dv_1}{dx}$$

0 B.C.

$$v_1(x) = \cancel{v_1(0)} + \frac{1}{EI}\int_0^x \left[M_A x + A_y \frac{x^2}{2}\right] dx$$

$$v_1(x) = \frac{1}{EI}\left[M_A \frac{x^2}{2} + A_y \frac{x^3}{6}\right]$$

B.C $v_1(L) = 0$

$$\cancel{\frac{1}{EI}}\left[M_A \frac{\cancel{L^2}}{\cancel{2}} + A_y \frac{L^{\cancel{3}}}{\cancel{6}_3}\right] = 0$$

$$M_A + A_y \frac{L}{3} = 0 \Rightarrow A_y = -\frac{3M_A}{L} \qquad (3)$$

Replace (3) in (1)

$$B_y = P - A_y = P + \frac{3M_A}{L} \qquad (4)$$

Replace (4) in (2)

$$-M_A + \left(P + \frac{3M_A}{L}\right)L - 2PL = 0$$

Solve $\boxed{M_A = \frac{PL}{2}}$ (5)

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Replace (5) in (4)

$$B_y = P + 3\left(\frac{PL}{2}\right) = \boxed{\frac{5}{2}P = B_y} \quad (6)$$

Replace (5) in (3)

$$A_y = -\frac{3PL}{2L} \qquad \boxed{A_y = -\frac{3P}{2}} \quad (7)$$

Continuity equations: $v_1(L) = v_2(L) = 0$

$\theta_1(L) = \theta_2(L)$

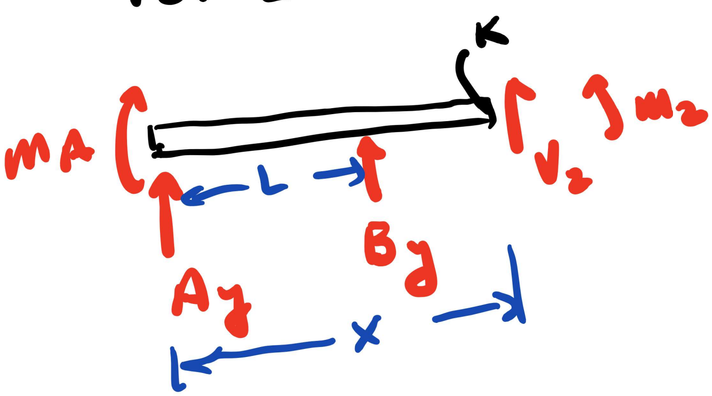
Using $\theta_1(x)$:

$$\theta_1(L) = \frac{1}{EI}\left[M_A L + A_y \frac{L^2}{2}\right] = \theta_B$$

Replacing M_A & A_y from (5) & (7)

$$\boxed{\theta_B = -\frac{PL^2}{4}}$$

For $L < x < 2L$



$$\Sigma M_K = M_2 - M_A - A_y - B_y(x-L) = 0$$

$$M_2(x) = M_A + A_y x + B_y(x-L)$$

$$M_2(x) = M_A + (A_y + B_y)x - B_y L$$

Replacing A_y, B_y, m_A

$$m_2(x) = P(x-2L) = EI\frac{d\theta_2}{dx}$$

$$\theta_2(x) = \underbrace{\theta_2(L)}_{\theta_B} + \frac{P}{EI}\int_L^x (x-2L)\,dx$$

$$= -\frac{PL^2}{4EI} + \frac{P}{EI}\left[\frac{1}{2}(x^2-L^2) - 2L(x-L)\right] = \frac{dv_2}{dx}$$

$$v_2(x) = \cancel{v_2(L)}^{\,0 \text{ B.C.}} + \int_L^x \theta_2(x)\,dx$$

$$v_2(x) = -\underbrace{\frac{PL^2}{4EI}}_{\theta_B}(x-L) - \frac{2PL}{EI}\frac{(x^2-L^2)}{2} + \frac{2PL^2}{EI}(x-L) + \frac{P}{2EI}\frac{(x^3-L^3)}{3} - \frac{PL^2}{2EI}(x-L)$$

Expanding with similar terms:

$$\boxed{v_2(x) = \frac{P}{EI}\left[\frac{x^3}{6} - Lx^2 + \frac{5}{4}xL^2 - \frac{5}{12}L^3\right]}$$

$$\boxed{v_1(x) = \frac{Px^2}{4EI}(L-x)}$$

Using indefinite integrals, different cuts were also acceptable

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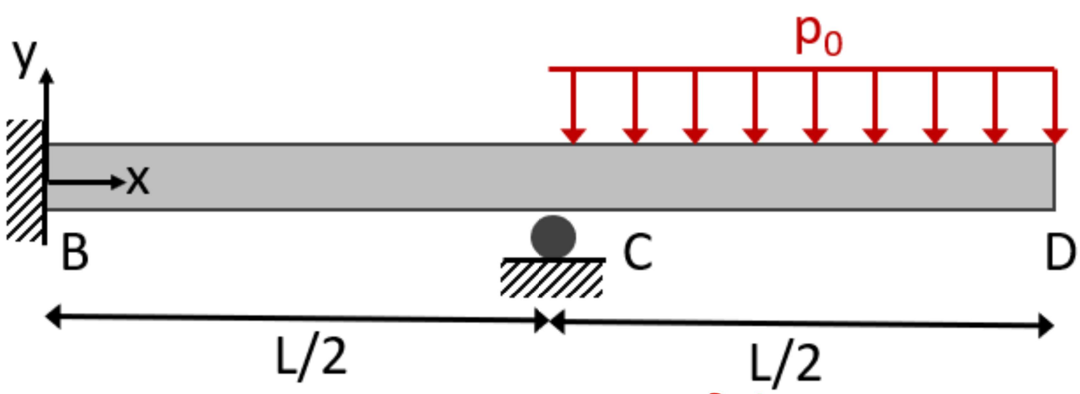
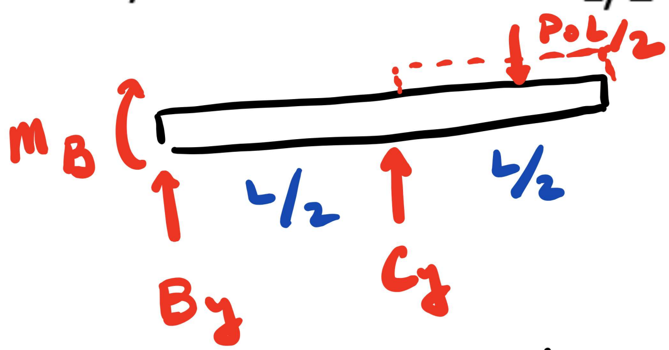
PROBLEM #2 (25 points)

A cantilever beam BCD has a distributed load p_0 acting between C and D.

(a) Draw a free body diagram and write the equilibrium equations.
(b) **Use the superposition principle** and the superposition tables provided to calculate the values of the reactions at B and C. Leave your answers in terms of p_0 and L.
(c) Draw the internal moment M(x) and shear force V(x) along the beam on the axes on the next page. Label the values of M(x) and V(x) at points B, C, and D.



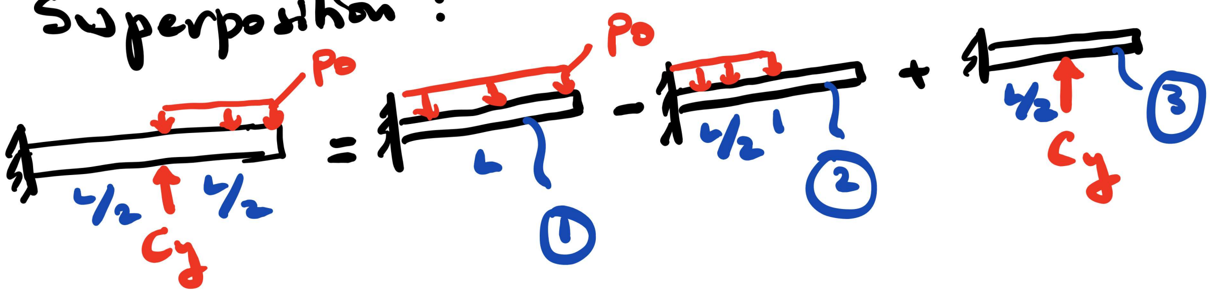
FBD :



Equilibrium : $\sum M_B = -M_B + C_y \frac{L}{2} - p_0 \frac{L}{2}\left(\frac{3L}{4}\right) = 0$ (1)

$\sum F_y = B_y + C_y - p_0 \frac{L}{2} = 0$ (2)

Superposition :



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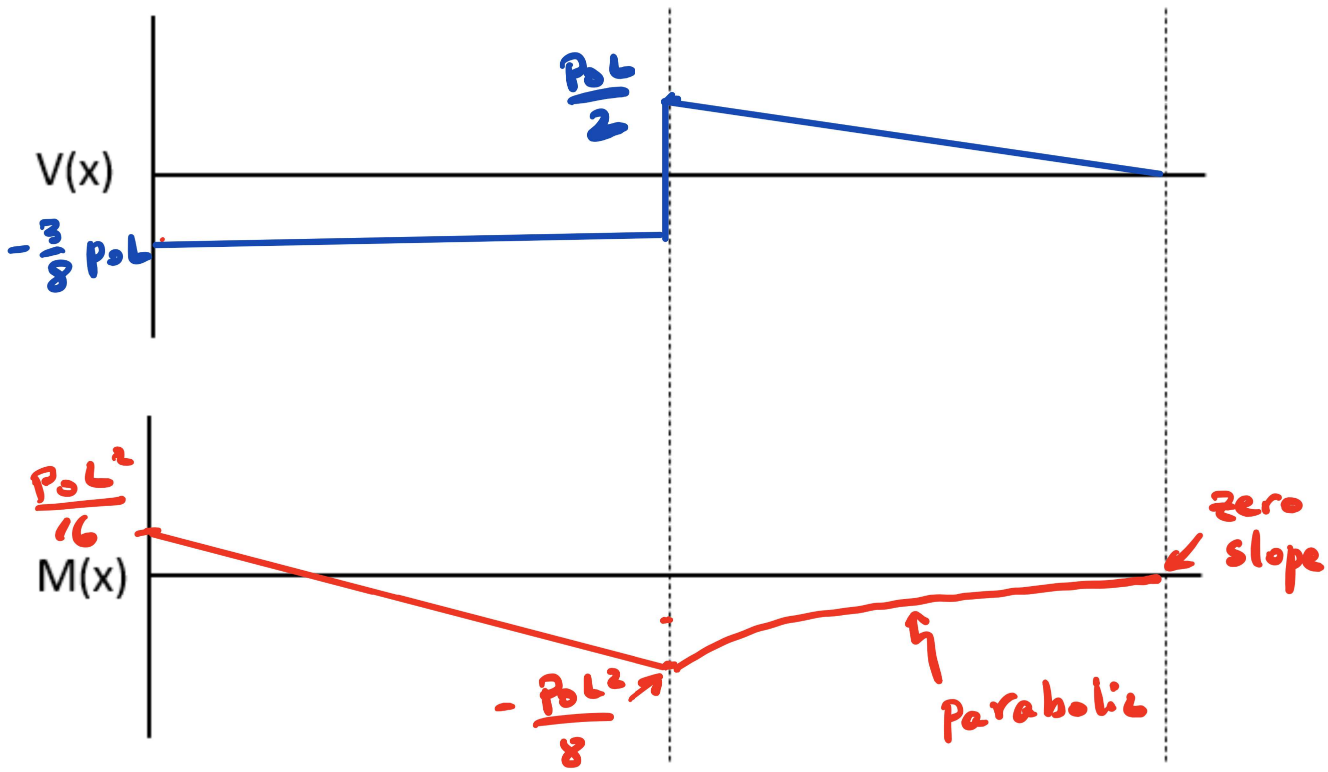

$$v(x) = v_1(x) - v_2(x) + v_3(x)$$

From superposition tables

$$v_1(x) = -\frac{p_0 x^2}{24EI}\left(6L^2 - 4Lx + x^2\right)$$

$$v_2(x) = -\frac{p_0 x^2}{24EI}\left[6\left(\frac{L}{2}\right)^2 - 4\left(\frac{L}{2}\right)x + x^2\right] \quad 0 < x < L/2$$

$$v_3(x) = \frac{C_y x^2}{6EI}\left[3\left(\frac{L}{2}\right) - x\right] \quad 0 < x < \frac{L}{2}$$

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B.C: $v\left(\frac{L}{2}\right)=0$

$$v\left(\frac{L}{2}\right)=\frac{-p_0\left(\frac{L}{2}\right)^2}{24EI}\left[6L^2-4L\left(\frac{L}{2}\right)+\frac{L^2}{4}\right]$$

$$+\frac{p_0\left(\frac{L}{2}\right)^2}{24EI}\left[6\frac{L^2}{4}-4\left(\frac{L^2}{4}\right)+\frac{L^2}{4}\right]$$

$$+\frac{C_y\left(\frac{L}{2}\right)^2}{6EI}\left[3\frac{L}{2}-\frac{L}{2}\right]=0$$

$$v\left(\frac{L}{2}\right)=-\frac{p_0L^2}{4}\left(\frac{17}{4}\right)+\frac{p_0L^2}{4}\left(\frac{3}{4}\right)+C_yL=0$$

$$\boxed{C_y=\frac{7}{8}p_0L}$$

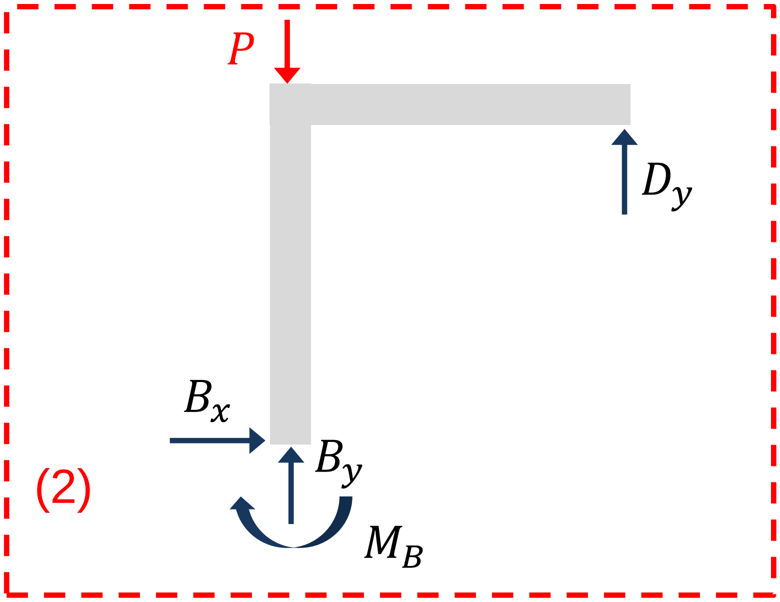
Replace C_y in eq'n (2): $\boxed{B_y=-\frac{3}{8}p_0L}$

Replace C_y & B_y in (1):

$$\boxed{M_B=\frac{p_0L^2}{16}}$$

Back to V & M diagrams

(a) FBD



Equilibrium

$$(\Sigma M)_B = -M_B + D_y L = 0$$

$$\Sigma F_x = B_x = 0$$

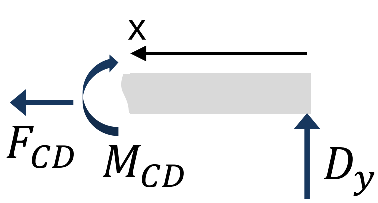
$$\Sigma F_y = B_y + D_y - P = 0$$

(4)

4 unknowns and 3 equations → need 1 redundant load.

(b) Solve for reactions

Use D_y as redundant



$$\Sigma M = 0 = -M_{CD} + xD_y$$

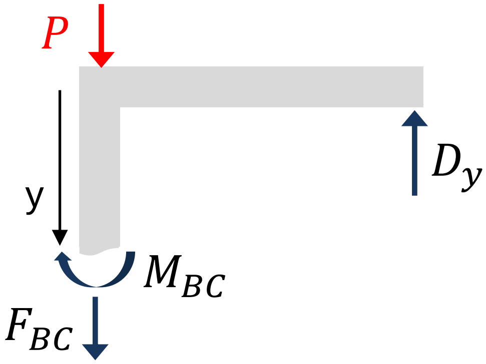
$$\rightarrow M_{CD}(x) = xD_y$$

(2)

$$\Sigma F_x = 0 = -F_{CD}$$

$$\rightarrow F_{CD}(x) = 0$$

(1)



$$\Sigma M = 0 = -M_{BC} + LD_y$$

$$\rightarrow M_{BC}(x) = LD_y$$

(2)

$$\Sigma F_y = 0 = -F_{BC} - P + D_y$$

$$\rightarrow F_{BC}(x) = D_y - P$$

(1)

$$U_{total} = \frac{1}{2EI}\int_0^L M_{BC}^2 dy + \frac{1}{2EA}\int_0^L F_{BC}^2 dy + \frac{1}{2EI}\int_0^L M_{CD}^2 dx + \frac{1}{2EA}\int_0^L F_{CD}^2 dx$$

(2)

$$\Delta_D = 0 = \frac{\delta U}{\delta D_y}$$

(2)

$$0 = \frac{1}{EI}\int_0^L M_{BC}\frac{\delta M_{BC}}{\delta D_y}dy + \frac{1}{EA}\int_0^L F_{BC}\frac{\delta F_{BC}}{\delta D_y}dy + \frac{1}{EI}\int_0^L M_{CD}\frac{\delta M_{CD}}{\delta D_y}dx$$

$$\frac{\delta M_{BC}}{\delta D_y} = L \qquad \frac{\delta F_{BC}}{\delta D_y} = 1 \qquad \frac{\delta M_{BC}}{\delta D_y} = x \qquad \frac{\delta F_{BC}}{\delta D_y} = 0$$

(2)

$$0 = \frac{1}{EI}\int_0^L D_y x^2 dy + \frac{1}{EA}\int_0^L (D_y - P)dy + \frac{1}{EI}\int_0^L D_y L^2 dy$$

(1)

$$0 = \frac{D_y L^3}{3EI} + \frac{D_y L^3}{EI} + \frac{(D_y - P)L}{EA} \longrightarrow D_y\left(\frac{4L^3}{3EI} + \frac{L}{EA}\right) = \frac{PL}{EA}$$

$$D_y = \frac{P}{\left(\frac{4AL^3}{3I} + 1\right)} \qquad B_y = P - \frac{P}{\left(\frac{4AL^3}{3I} + 1\right)} \qquad M_B = \frac{PL}{\left(\frac{4AL^3}{3I} + 1\right)}$$

(1)

(c) Solve for displacement at C

$$M_{CD}(x) = \frac{xP}{\left(\frac{4AL^3}{3I}+1\right)} = \frac{xP}{K} \qquad M_{BC}(x) = \frac{LP}{\left(\frac{4AL^3}{3I}+1\right)} = \frac{LP}{K}$$

$$F_{BC}(x) = \frac{P}{\left(\frac{4AL^3}{3I}+1\right)} - P = \frac{P}{K} - P \qquad K = \left(\frac{4AL^3}{3I}+1\right) \quad (1)$$

$$\Delta_C = \frac{\delta U}{\delta P} \quad (2)$$

$$\frac{\delta M_{BC}}{\delta P} = \frac{L}{K} \qquad \frac{\delta F_{BC}}{\delta D_y} = \frac{1}{K} - 1 \qquad \frac{\delta M_{BC}}{\delta D_y} = \frac{x}{K}$$

$$\Delta_C = \frac{1}{EI}\int_0^L P\frac{x^2}{K^2}dy + \frac{1}{EI}\int_0^L P\frac{L^2}{K^2}dy + \frac{1}{EA}\int_0^L \left(\frac{P}{K} - P\right)\left(\frac{1}{K} - 1\right)dy$$

$$\Delta_C = \frac{4PL^3}{3EIK^2} + \frac{PL}{EA}\left(\frac{1}{K^2} - \frac{2}{K} + 1\right) \qquad K = \left(\frac{4AL^3}{3I}+1\right) \quad (1)$$

Most Common Errors

1. Internal reactions need to be in terms of only the redundant load before taking the partial derivatives (in this example solution, M_{BC}, M_{CD}, and F_{CD} were a function of only D_y and were not a function of B_y or M_B).
2. When solving for the displacement, need to substitute the solved values of the reactions ($D_y = P/K$) into the equations for the external reactions ($M_{BC} = LP/K$) before taking the partial derivatives with respect to P.

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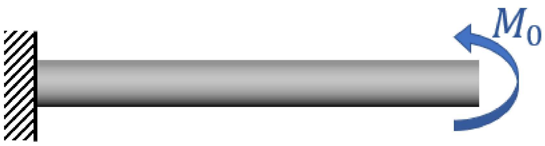
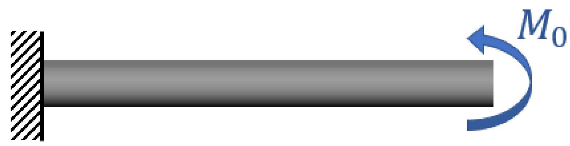
PROBLEM #4 – PART A (6 points)

Beam (i) and (ii) are identical cylindrical beams except that beam (i) is made of steel and beam (ii) is made of aluminum. $E_{steel} > E_{aluminum}$.



Beam (a) Steel



Beam (b) Aluminum

(a) Circle the correct relationship between the maximum shear stresses in the two beams (**1 point**).

$|\tau_{max,a}| < |\tau_{max,b}|$

$|\tau_{max,a}| = |\tau_{max,b}| = 0$

$|\tau_{max,a}| = |\tau_{max,b}| \neq 0$

$|\tau_{max,a}| > |\tau_{max,b}|$

(b) Circle the correct relationship between the maximum shear stresses in the two beams (**1 point**).

$|\sigma_{max,a}| < |\sigma_{max,b}|$

$|\sigma_{max,a}| = |\sigma_{max,b}| = 0$

$|\sigma_{max,a}| = |\sigma_{max,b}| \neq 0$

$|\sigma_{max,a}| > |\sigma_{max,b}|$

(c) Circle the correct relationship between the maximum deflection $v(x)$ in the two beams (**1 point**).

$|v_{max,a}| < |v_{max,b}|$

$|v_{max,a}| = |v_{max,b}| = 0$

$|v_{max,a}| = |v_{max,b}| \neq 0$

$|v_{max,a}| > |v_{max,b}|$

$$v(L) = \frac{M_0 L^2}{2EI}$$

larger E $\Rightarrow$ smaller v.

(d) The diameter of the original beams is D. If the diameter is doubled to 2D, how will the new deflection of the new beam (v^*_{max}) with diameter of 2D compare to the deflection of the original beam (v_{max}) with diameter of D (**3 points**):

$v_{max} = v^*_{max}$

$v_{max} = 2v^*_{max}$

$v_{max} = 4v^*_{max}$

$v_{max} = 8v^*_{max}$

$v_{max} = 16v^*_{max}$

$$I = \frac{\pi}{4}\left(\frac{D}{2}\right)^4 = \frac{\pi D^4}{64}$$

$$I^* = \frac{\pi}{4}(D)^4 = \frac{\pi D^4}{4} = 16I$$

$$v(L) = \frac{M_0 L^2}{2EI} \qquad v^*(L) = \frac{M_0 L^2}{2E(16I)} = \frac{v}{16}$$

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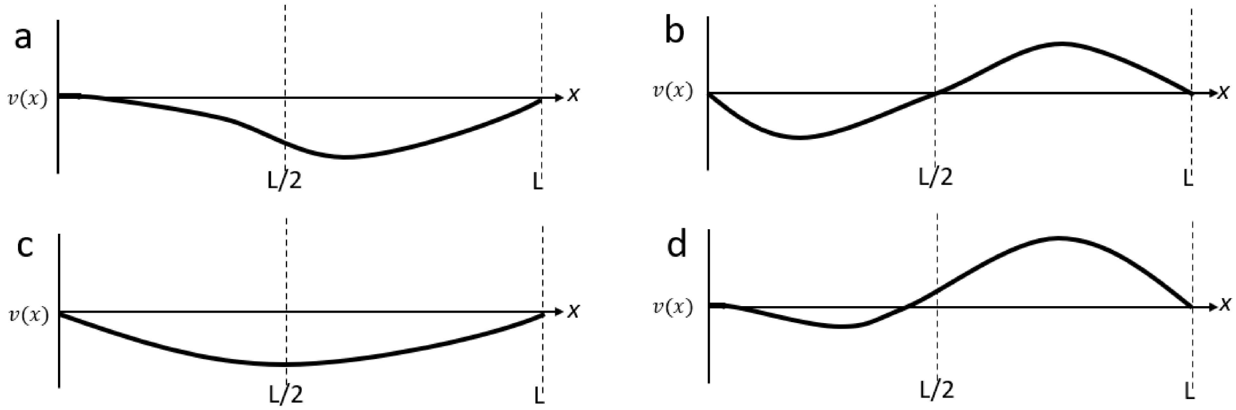
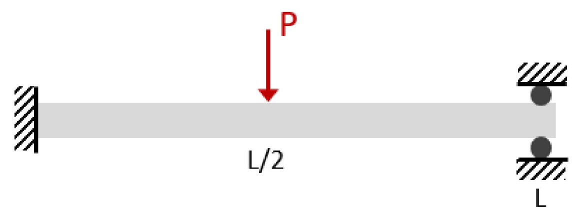
PROBLEM 4 – PART B (6 points)

Figures a-d indicate the deflection curve along four different beams.

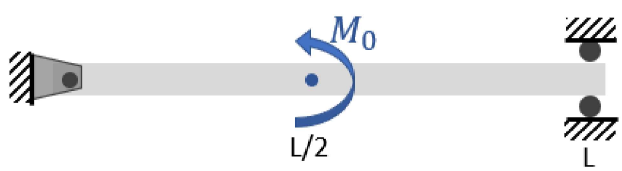


(i) Circle the deflection curve that corresponds to the given beam and loading conditions (**2 points**):



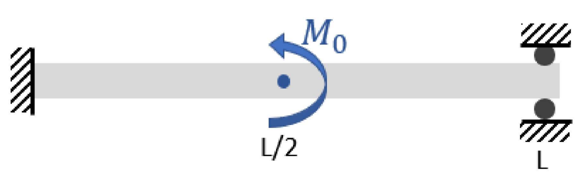
a b c d

(ii) Circle the deflection curve that corresponds to the given beam and loading conditions (**2 points**):



a b c d

(iii) Circle the deflection curve that corresponds to the given beam and loading conditions (**2 points**):



a b c d

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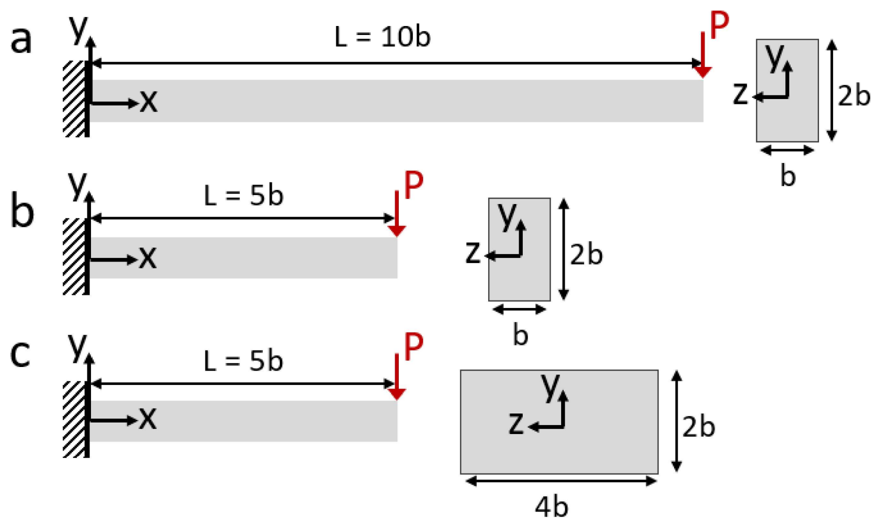
PROBLEM 4 – PART C (2 points)

Based on the assumptions used when deriving the equation for shear stress on a beam cross-section ($\tau = VQ/It$), choose the correct ranking for the accuracy of the shear stress predicted by this equation for the three beams shown below:



	Option 1	Option 2	Option 3	Option 4	Option 5
Most accurate	a	a	b	c	All have the same accuracy
	b	c	a	b	
Least accurate	c	b	c	a	

See Chapter 10, page 20

Assumptions are valid for:

- Beams that are long (L) compared to their depth (z-thickness)
- Beams that are thin in the depth (z-thickness) direction.

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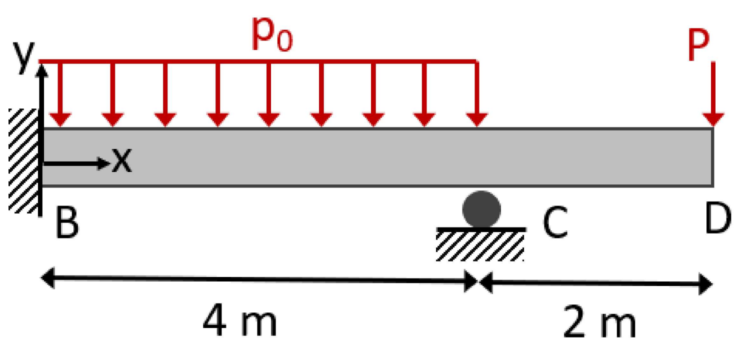
PROBLEM 4 – PART D (6 points)

A beam is loaded with a distributed load from 0 to 4 m and a point load at 6 m.



Circle the value(s) that will be zero at x = 0m (**2 points**):

V(0) M(0) θ(0) v(0)

Circle the value(s) that will be zero at x = 4m (**2 points**):

V(4) M(4) θ(4) v(4)

Circle the value(s) that will be zero at x = 6m (**2 points**):

V(6) M(6) θ(6) v(6)

also gave the mark if this is circled.

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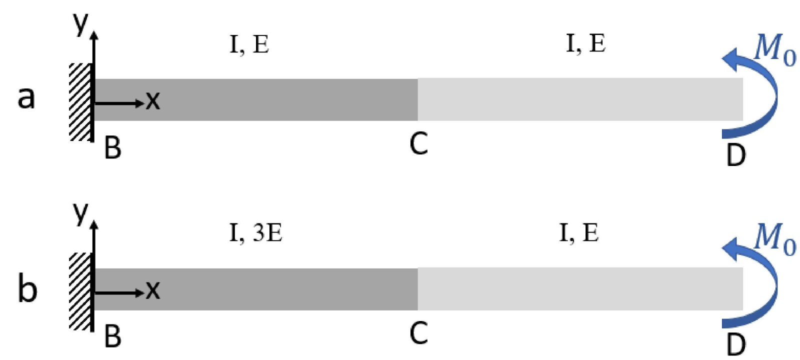
PROBLEM 4 – PART E (5 points)

A simple cantilever is composed of two sections with an applied moment at the end.



$$U = U_1 + U_2 = \frac{1}{2EI}\int_0^{L/2} M_1^2\,dx + \frac{1}{2EI}\int_{L/2}^{L} M_2^2\,dx.$$

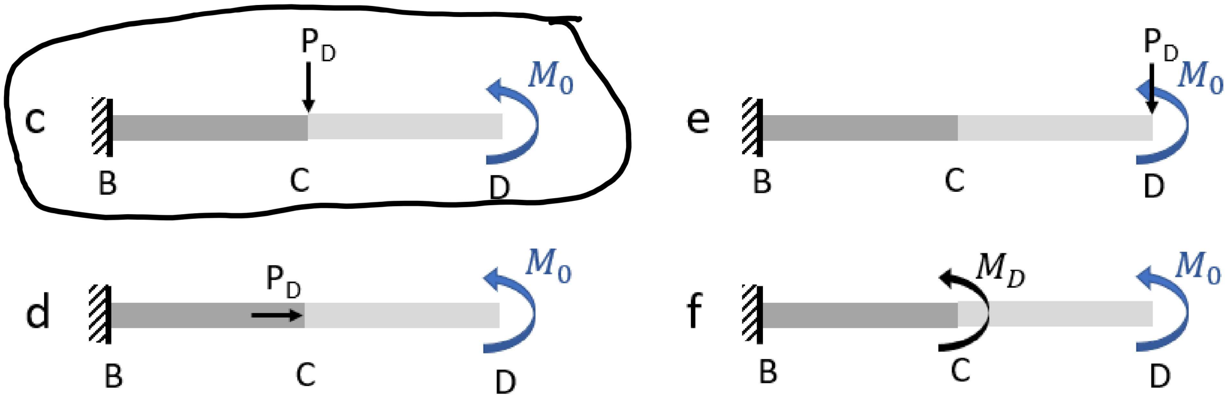
(i) **(3 points)** In beam (a), the two sections both have the same Young's moduli of E. In beam (b), one of the sections has a Young's modulus of 3E, while one has a Young's modulus of E. How does the total strain energy of these two beams compare?:

- $U_{total,a} > U_{total,b}$
- $U_{total,a} = U_{total,b}$
- $U_{total,a} < U_{total,b}$

$$U_a = \frac{1}{2EI} M_0^2\left(\frac{L}{2}\right) + \frac{1}{2EI} M_0^2\left(\frac{L}{2}\right) = \frac{M_0^2 L}{2EI}$$

$$U_b = \frac{1}{2(3E)I} M_0^2\left(\frac{L}{2}\right) + \frac{1}{2EI} M_0^2\left(\frac{L}{2}\right) = \frac{M_0^2 L}{3EI}$$

(ii) **(2 points)** Circle the loading condition below (c to f) that would be used if we want to calculate the deflection at point C in the y-direction.



Need dummy load in y-direction at C.