ECE 201 – Fall 2014
Exam 3
November 20, 2014

Section (circle below)

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Name ___________________________ Solutions ___________________________ PUID ___________________________

Instructions

1. DO NOT START UNTIL TOLD TO DO SO.
2. Write your name, section, professor, and student ID# on your Scantron sheet. We may check PUIDs.
3. This is a CLOSED BOOKS and CLOSED NOTES exam.
4. The use of a simple scientific calculator without memory or communication capabilities (such as TI-30Xa or TI-30X IIS) is allowed, but not necessary.
5. If extra paper is needed, use the back of test pages.
6. Cheating will not be tolerated. Cheating in this exam will result in, at the minimum, an F grade for the course. In particular, continuing to write after the exam time is up is regarded as cheating.
7. If you cannot solve a question, be sure to look at the other ones, and come back to it if time permits.
8. Exam #3 provides evidence for satisfaction of this ECE 20100 Learning Objective:
   iii) An ability to analyze 2nd order linear circuits with sources and/or passive elements.
   The minimum score needed to satisfy this objective will be posted on Blackboard after the exam has been graded. Remediation options will be posted in Blackboard if you fail to satisfy any of the course outcomes.
9. The last two pages of the exam contain potentially useful formulas.

Honor Pledge: I have neither given nor received unauthorized assistance on this exam

Signature: ___________________________
Find R so that the circuit is critically damped.

1. R = 0 Ohm
2. R = 0.25 Ohm
3. R = 0.5 Ohm
4. R = 1 Ohm
5. R = 2 Ohm
6. R = 4 Ohm
7. R = 10 Ohm
8. R = 50 Ohm
9. It cannot be found with the given data
10. None of the above

When critically damped, the coefficients of the characteristic equation must satisfy:
\[ b^2 - 4c = 0 \]

To find b and c for \( \Theta \), analyze the source-free circuit:

This is a parallel RLC with:
\[ R_{eq} = R, \ L = 1F, \ C = 1H \]

\[ b = \frac{1}{RL}, \ c = \frac{1}{LC} \] from equation sheet

So,
\[ b = \frac{1}{R}, \ c = 1 \]

Plug (2) into (1) and solve for R:
\[ \frac{1}{R^2} - 4 = 0 \]
\[ R = \sqrt{4} = 2 \]
For the same circuit from the previous question (copied above for your convenience), find $i_L(\infty)$, the current through the inductor once all the transient behavior has died out. The value of the resistance $R$ may be different from the solution of the previous question.

1. $i_L(\infty) = -2$ Amp, regardless of $R$
2. $i_L(\infty) = -1.5$ Amp, regardless of $R$
3. $i_L(\infty) = -\frac{1}{2R}$ Amp
4. $i_L(\infty) = 0$ Amp, regardless of $R$
5. $i_L(\infty) = 1$ Amp, regardless of $R$
6. $i_L(\infty) = 2$ Amp, regardless of $R$
7. $i_L(\infty) = 2 - \frac{3}{R}$ Amp
8. $i_L(\infty) = 3 - \frac{1}{R}$ Amp
9. $i_L(\infty) = 2 + \frac{R}{2}$ Amp
10. $i_L(\infty) = \frac{5}{R}$ Amp

- Transient behavior dies out for $t \to \infty$
  Thus the capacitor looks like an open & the inductor looks like a short:

- We can use superposition to find $i_L(\infty)$.
  Due to the 2A source (zero-out 3V source):

- Due to 3V source (zero-out 2A source):
  Superposition of $\square$ & $\square$:
  $i_L(\infty) = i_L^{(2)}(\infty) + i_L^{(3)}(\infty) = 2 - \frac{3}{R}$ Amp
Question 3:

For the same circuit from the previous question (copied above for your convenience) find the derivative of the current through the inductor at $t = 0^+$.

1. $i_L'(0^+) = -5 \text{ Amp/sec}$
2. $i_L'(0^+) = -3 \text{ Amp/sec}$
3. $i_L'(0^+) = -1 \text{ Amp/sec}$
4. $i_L'(0^+) = 0 \text{ Amp/sec}$
5. $i_L'(0^+) = 1 \text{ Amp/sec}$
6. $i_L'(0^+) = 3 \text{ Amp/sec}$
7. $i_L'(0^+) = 5 \text{ Amp/sec}$
8. $i_L'(0^+) = 10 \text{ Amp/sec}$

9. The current through an inductor is continuous, so its derivative is always zero.
10. The answer depends on the value of resistance $R$. 

From definition of inductance:

$$v_c(0^-) = L \frac{di_c}{dt} \bigg|_{t=0^+}$$

$$i_c'(0^+) = \frac{v_c(0^+)}{L} = v_c(0^+) \tag{0}$$

$i_c$ and $v_c$ are continuous, so:

$$v_c(0^+) = v_c(0^-)$$

$$i_c(0^+) = i_c(0^-)$$

Because no sources are on for $t \leq 0$

$$v_c(0^+) = v_c(0^-) = 0 \text{ V}$$

$$i_c(0^+) = i_c(0^-) = 0 \text{ A}$$

$\therefore v_c(0^+) = V_B \cdot 0 = V_B$

$\therefore V_B = V_A - 3\bar{V}$

Voltage source

$\therefore v_A = 0 + 0 = 0 \text{ V}$

$\therefore v_c(0^+) = -3 \text{ V}$

Plug $\circled{3}$ into $\circled{2}$ then into $\circled{1}$

$V_B = -3 \text{ V}$

Plug $\circled{0}$ into $\circled{\circled{5}}$:

$$i_L(0^+) = -3 \text{ A}$$
In the above circuit, the capacitor voltage at time \( t=0 \) is 10V. Which statement is true for any non-negative (i.e., zero, positive, or infinite) value of \( R \)? Mark only one answer. (Hint: Consider the range of the equivalent resistance first.)

(1) The circuit is overdamped

(2) The circuit is critically damped

(3) The circuit is underdamped with angular frequency equal to \( \sqrt{50} \) rad/sec

(4) The circuit is undamped (oscillator) with angular frequency equal to \( \sqrt{50} \) rad/sec

(5) The circuit can be overdamped or critically damped, depending on \( R \)

(6) The circuit oscillates (underdamped or undamped) with angular frequency depending on \( R \)

(7) Since there are no sources, all the voltages in the circuit remain constant

(8) The capacitor behaves like an open circuit, so no current can flow through it or the inductor

(9) The inductor behaves like a short circuit, so there can be no voltage drop across it

(10) None of the above

* Because we have series RLC:

\[
\frac{b}{L} = \frac{1}{L} = \frac{1}{2(10)^2 \times 10^{-3}} = \frac{1}{2} \times 10^3 = 0.05 \times 10^5 = 50
\]

\[
\therefore b^2 - 4c = \frac{1}{4} \text{Req}^2 - 200 \rightarrow \max \{ b^2 - 4c \} = \frac{1}{4} \text{Req}^2 - 200 \geq 25 - 200 = -175 < 0
\]

\[
\min \{ b^2 - 4c \} = \frac{1}{4} \text{Req}^2 - 200 = 0 - 200 = -200 < 0
\]

* Angular frequency is:

\[
\omega_d = \sqrt{\frac{b^2 - 4c}{L}} = \sqrt{\frac{200 - \frac{1}{4} \text{Req}^2}{2}} \Rightarrow \omega_d \text{ dependent on Req}
\]

\[ \therefore \text{always underdamped} \]

\[ \text{undamped if} \quad \text{Req} = 0 \]
Question 5:

In the circuit above, the switch has been closed for a long time and opens at \( t=0 \). Find \( V_c(t) \) for \( t>0 \) (in Volts).

1) For \( t>0 \):
   - Series RLC:
     \[ b = \frac{R}{L} = \frac{2 \times 10^3}{1} = 2000 \]
     \[ c = \frac{1}{LC} = \frac{1}{1 \times (4 \times 10^{-6})} = \frac{4}{10^4} \]
     \[ a^2 - 4c = (2 \times 10^3)^2 - 4 \left( \frac{4}{10^4} \right) = 4 \times 10^6 - 3 \times 10^3 = 1 \times 10^3 > 0 \]
     \( \Rightarrow \text{OVERDAMPED} \)

2) \( V_c(t) = 8 + \exp(-500t) + \exp(-1000t) \)

3) \( V_c(t) = 8 \exp(-1000t) \)

4) \( V_c(t) = 10 - 2.25 \exp(-500t) + 0.25 \exp(-1500t) \)

5) \( V_c(t) = (8 - 2t) \exp(-1000t) \)

6) \( V_c(t) = 10 + (2 - 2t) \exp(-1000t) \)

7) \( V_c(t) = 8 + 2t \exp(-1500t) \)

8) \( V_c(t) = 10 - 2 \cos(1500t) \exp(-500t) \)

9) \( V_c(t) = 8 + 3 \cos(1000t) - \sin(100t) \)

II) Because it is over-damped, solution form for \( V_c(t) \) is:
   \[ V_c(t) = V_c(0) + K_1 e^{s_1t} + K_2 e^{s_2t} \]

Finding unknowns...
- First \( s_1 \) and \( s_2 \):
  \[ s_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2} = -1 \times 10^3 \pm \frac{1}{2} \sqrt{1 \times 10^6 - 3 \times 10^3} \]
  
- Finding \( V_c(0) \): For \( t>0 \), short \( L \) and open \( C \)
  \[ V_c(0) = 10V \]

III) Finding \( K_1 \) and \( K_2 \):

- Need initial conditions:
  \[ \Rightarrow V_c(0^+) = V_c(0^-) \]

\( t=0^- \) circuit looks like:

\[ \begin{aligned}
10V & \quad \text{V}^+ \\
\text{V}^- & \quad 2k\Omega \\
\text{V}_c(0^-) & \quad 8 \quad \text{V}^- \\
\end{aligned} \]

From v-division, \( \text{V}_c(0^-) = \frac{10}{8} \times B \text{ V} \)

\[ \Rightarrow \text{V}_c(0^-) = 8 \text{ V} \]

- Plug initial conditions into \( V_c(t) \) and \( V_c(t) \) solution for \( t=0^- \):
  \[ \begin{align*}
\text{V}_c(0^-) &= 8 = 10 + K_1 + K_2 \\
\text{V}_c(0^-) &= 750 = -500k_1 - 1500k_2
\end{align*} \]

Solve:
\[ \begin{align*}
500k_1 + 1500k_2 &= 8 \\
750 &= -500k_1 - 1500k_2
\end{align*} \]

\[ \Rightarrow k_1 = 0.25, k_2 = \frac{1}{2} \]

\[ \Rightarrow \text{plug into } (1): \quad 8 = 10 + 0.25 + 0.25 \]

\[ \Rightarrow k_1 = -2.25 \]

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Question 6:

Find R so that \( \frac{V_{out}}{V_{in}} = 4 \). Assume that the Op-Amp is ideal.

(1) \( R = 0 \) Ohms
(2) \( R = 0.1 \) kOhms
(3) \( R = 0.25 \) kOhms
(4) \( R = 0.5 \) kOhms
(5) \( R = 0.75 \) kOhms
(6) \( R = 1 \) kOhms
(7) \( R = 5 \) kOhms
(8) \( R = 7.5 \) kOhms
(9) \( R = 12 \) kOhms
(10) None of the above

\[ \text{Fast way:} \]
This is an inverting amplifier
\[ \implies \frac{V_{out}}{V_{in}} = -\frac{R_2}{R_1} \]
\[ \implies \frac{V_{out}}{V_{in}} = -\frac{2000}{10} = -20 \]
\[ R = \frac{3}{4} \times 10^2 = 0.75 \text{ k} \Omega \]

\[ \text{Long way:} \]
Always begin ideal Op-Amp problems with:
\[ V_p = V_o \]
\[ I_p = 0 \]
\[ I_o = 0 \]

Since \( I_p = 0 \), no current flows through load resistor and, by Ohm's law, \( V_p = 0 \).
\[ V_o = 0 \implies V_o = V_{in} - 0 = V_{in} \]

From Ohm's law, then,
\[ i_1 = \frac{V_o}{R_1} = \frac{V_{in}}{10} \]

Due to KCL and fact that \( i_1 = 0 \) \( \implies i_2 = i_1 \)

From Ohm's law then,
\[ V_2 = i_2 R_2 = i_1 (3000) = \frac{V_{in}}{10} \]

\( V_{out} \) is defined found as:
\[ V_{out} = V_o - V_2 = V_{in} - \frac{3000}{10} \]
\[ \implies \frac{V_{out}}{V_{in}} = -\frac{3000}{10} \]

If we let \( \frac{V_{out}}{V_{in}} = -4 \) (from problem statement):
\[ -\frac{3000}{10} = -4 \]
\[ R = 0.75 \text{ k} \Omega \]
Question 7:

Assume that the Op-Amp in the circuit below is ideal. Find the output current $I_{out}$ in the following circuit:

1. Start all ideal op-amp problems with:
   
   $V_{+} = V_{-}$
   
   $i_{+} = 0$
   
   $i_{-} = 0$

2. Since $i_{+} = 0$, due to $I_{in}$'s law,
   
   $V_{-} = V_{+} = V_{A}$

3. By definition of the $V$-source:
   
   $V_{A} = V_{A} + 10 \Omega$

4. By definition of $V_{i}$:
   
   $V_{i} = V_{+} - V_{-} = V_{A} - V_{A} = V_{A} + 10 - V_{A} = 10 \text{ V}$

5. From $I_{in}$ law:
   
   $i_{i} = \frac{V_{i}}{1000} = 0.01 \text{ A} = 10 \text{ mA}$

6. And:
   
   $V_{A} = V_{3} = i_{3} \times 1000 = i_{1} \times 1000 = 10 \text{ V}$

7. Because $i_{+} = 0$, $i_{2} = i_{1}$ and by $I_{in}$'s law:
   
   $V_{2} = i_{2} \times 5 \text{ K} = i_{1} \times (5 \times 10^{3}) = 50 \text{ V}$

8. $V_{c}$, then, is:
   
   $V_{c} = V_{+} + V_{2} = 20 + 50 = 70 \text{ V}$

9. From $I_{in}$'s law:
   
   $I_{out} = \frac{V_{c} - 0}{2 \times 10^{3}} = 35 \text{ mA}$

10. None of the above
Question 8:

Assume ideal op-amp. The Thevenin equivalent of the circuit on the left when looking into terminals A and B is:

\[ V_{TH} = V_T = V_A - V_B \]

- Need both \( V_{TH} \) and \( R_{TH} \)

1. \( V_s = 2V, R_{th} = 6\Omega \)
2. \( V_s = 2V, R_{th} = 6\Omega \)
3. \( V_s = -2V, R_{th} = 6\Omega \)
4. \( V_s = -2V, R_{th} = 6\Omega \)
5. \( V_s = 3V, R_{th} = 10\Omega \)
6. \( V_s = 3V, R_{th} = 10\Omega \)
7. \( V_s = 5V, R_{th} = -25\Omega \)
8. \( V_s = 2V, R_{th} = 15\Omega \)

**Method 2:** Find \( I_{SC} \) and then

\[ R_{TH} = \frac{V_{TH}}{I_{SC}} \]

To find \( I_{SC} \):

- Because \( V_A = V_B = 5V \)
- KCL at C:
  \[ I_{SC} = i_1 + i_2 \]
- KVL for \( i_1 \) and \( i_2 \):
  \[ i_1 = \frac{V_0 - 5}{10} = \frac{0 - 5}{10} = -\frac{1}{2} \text{A} \]
  \[ i_2 = \frac{V_0 - 5}{15} = \frac{0 - 5}{15} = -\frac{1}{3} \text{A} \]
- \( I_{SC} = 0 + \frac{1}{2} = \frac{1}{3} \text{A} \)
- Plug \( V_0 \) and \( I_{SC} \) into 0:
  \[ R_{TH} = \frac{V_0}{I_{SC}} = \frac{5}{\frac{1}{3}} = 15 \text{\Omega} \]

- (Two methods)

Method 1: Apply test current source with all other independent sources zeroed, then \( R_{TH} = \frac{V_I}{I_T} \)

Let \( I_T = 1 \text{Amp} \)

\[ V_T = V_T = V_C = V_C = 0 - V_C = V_C = V_C \]

\[ V_T = V_C = 6 \text{V} \]

- 10\( \Omega \) and 15\( \Omega \) are in parallel:
  \[ 10 \| 15 = \frac{10 \times 15}{10 + 15} = \frac{150}{25} = 6 \text{\Omega} \]
- From a's law:
  \[ 0 - V_C = I_T \times 15 \text{V} \]
  \[ -V_C = 6 \text{V} \]
- So, from 2:
  \[ V_T = -V_C = 6 \text{V} \]
  \[ R_{TH} = \frac{V_I}{I_T} = \frac{6 \text{V}}{1 \text{A}} = 6 \text{\Omega} \]
Question 9:

Assume ideal op-amp. The initial voltage of the 0.1F capacitor at time 0 is 0V. Find Vout(t) in Volts.

Quick way: This is an integrator. Therefore

\[ V_{out}(t) = V_{out}(0) - \frac{1}{RC} \int_{0}^{t} v_{in}(\tau) d\tau \]

\[ = 0 - \left[ 5\cos(\tau) \right]_{\tau=0}^{\tau=t} = -5\sin t \]

(1) -0.5 sin(t)
(2) 0.5 sin(t)
(3) -0.5 cos(t)
(4) 0, 5 cos(t)
(5) -5 sin(t)
(6) 5 sin(t)
(7) -5 cos(t)
(8) 5 cos(t)

Long way: Begin ideal op-amp problem with:

\[ V_c = v_c = 0 \]
\[ i_c = 0 \]

Since \( v_c = v_c = 0 \), by u's law:

\[ i_1 = \frac{v_1}{10} = \frac{v_{in} - 0}{10} = \frac{v_{in}}{10} \]

Because \( i_c = 0 \),

\[ i_c = i_1 = \frac{v_{in}}{10} \]

From our definition of capacitance:

\[ i_c = C \frac{dv_c}{dt} \Rightarrow \frac{dv_c}{dt} = \frac{i_c}{C} = \frac{v_{in}}{10C} \]

\[ \Rightarrow v_c(t) = v_c(0) + \int_{0}^{t} v_{in}(\tau) d\tau = \int_{0}^{t} 5\cos(\tau) d\tau \]

\[ = 5\sin t \bigg|_{\tau=0}^{\tau=t} = 5\sin t \]

From how we annotated \( v_c \),

\[ v_c = v_c = 0 - V_{out} \]

\[ \therefore V_{out}(t) = -v_c(t) = -5\sin t \]
Question 10:

In the circuit below, \( V_{in}(t) = 5 \cos(120t) \) V, \( V_{1}(t) = 2\sqrt{2} \cos(120t + 45^\circ) \) V. Find the phasor for the voltage \( V_2 \) in Volts. Assume sinusoidal steady-state. (Hint: Be careful of the reference directions of \( V_1 \) and \( V_2 \).)

\[ \begin{align*}
\text{(1) } & 3 - j2 \\
\text{(2) } & 3 \cos(120t) - j2 \sin(120t) \\
\text{(3) } & 3 + j2 \\
\text{(4) } & 3 \cos(120t) + j2 \sin(120t) \\
\text{(5) } & 7 - j2 \\
\text{(6) } & 7 \cos(120t) - j2 \sin(120t) \\
\text{(7) } & 7 + j2 \\
\text{(8) } & 7 \cos(120t) + j2 \sin(120t) \\
\end{align*} \]

To transform between time and phasor domain:
\[ x(t) = x_0 \cos(\omega t + \phi) \leftrightarrow \tilde{x} = x_0 e^{j\phi} \]

So in phasor domain:
\[ \begin{align*}
\tilde{V}_{in} & = 5 e^{j0} = 5 \text{ V} \\
\tilde{V}_{1} & = 2\sqrt{2} e^{j45^\circ \text{ V}} \\
\text{with } \omega & = 120 \text{ rad/s}
\end{align*} \]

Performing KVL:
\[ \tilde{V}_2 - \tilde{V}_1 - \tilde{V}_{in} = 0 \]
\[ \therefore \tilde{V}_2 = \tilde{V}_{in} + \tilde{V}_1 = 5 e^{j0} + 2\sqrt{2} e^{j45^\circ} = 5 + 2\sqrt{2} e^{j45^\circ} \]

Important: Both \( V_{in}(t) \) and \( V_1(t) \) are at SAME frequency, so we can do computations on their phasors together.

Can't add phasors in polar form. Must transform to cartesian form.

\[ \begin{align*}
\tilde{V}_{in} & = 5 + 0i \\
\tilde{V}_1 & = 2 + 2j \\
\tilde{V}_2 & = 2 + j2
\end{align*} \]

Plugging into \( 0 \):
\[ V_2 = V_{in} + V_1 = 5 + j0 + 2 + j2 = 7 + j2 \text{ V} \]
Question 11:

In the circuit below, find the phasor for the current $I_L$ through the 0.5H inductor (in Amps).
Assume sinusoidal steady-state.

In phasor domain:

\[ I_S = 6 \, e^{j0} \, A \quad \text{because } \omega L \cos (\omega t + \phi) \Rightarrow \tilde{X} = A e^{j\phi} \]
with \( \omega = 10 \, \text{rad/s} \)

- Impedances of elements are:
  
  \[ Z_C = \frac{1}{j\omega C} = \frac{1}{j10(0.01)} = \frac{1}{j10} = -j10 \]
  
  \[ Z_L = j\omega L = j10(0.5) = j5 \]
  
  \[ Z_R = R = 10 \]

- Admittances therefore are:
  
  \[ Y_C = \frac{1}{Z_C} = 0 \, \frac{1}{10} \]
  
  \[ Y_L = \frac{1}{j\omega L} = \frac{1}{j5} \times \frac{j}{j} = -\frac{j5}{25} = -\frac{j}{5} \]
  
  \[ Y_R = \frac{1}{Z_R} = \frac{1}{10} \]

- $I_L$ can be found by current division:

  \[ I_L = I_S \frac{Y_C}{Y_{eq}} = G \frac{-j5}{j10 - j5 - \frac{1}{10}} = G \frac{-j5}{j10 - j5 - \frac{1}{10}} \times \frac{10}{10} = 6 \frac{-2j}{1-j} \]

- Change denominator to polar form:

  \[ \frac{-2j}{1-j} \]

- Numerator:

  \[ -2j \]

- Denominator:

  \[ 2e^{-j90^\circ} \]

- \[ I_L = 6 \frac{ze^{j90^\circ}}{\sqrt{2} e^{-j45^\circ}} = 6 \sqrt{2} e^{j45^\circ} \]

- $\tilde{X} = 6 - j6$ A

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Question 12:

Assume sinusoidal steady state. The phasor voltage $V$ and phasor current $I$ for the device on the left are shown on the right. The angle between the two phasors is $60^\circ$. The magnitudes of the two phasors are 2 and 6, respectively, as shown in the figure on the right. Find the impedance $Z$ of the device in Ohms. (*Hint: $\cos 60^\circ = \frac{1}{2}$, $\sin 60^\circ = \frac{\sqrt{3}}{2}$.*)

\[ (1) \quad -\frac{1}{6} - j\frac{\sqrt{3}}{6} \]
\[ (2) \quad -\frac{1}{6} + j\frac{\sqrt{3}}{6} \]
\[ (3) \quad \frac{1}{6} - j\frac{\sqrt{3}}{6} \]
\[ (4) \quad \frac{1}{6} + j\frac{\sqrt{3}}{6} \]
\[ (5) \quad \frac{3}{2} - j\frac{3\sqrt{3}}{2} \]
\[ (6) \quad \frac{3}{2} + j\frac{3\sqrt{3}}{2} \]
\[ (7) \quad \frac{3}{2} - j\frac{3\sqrt{3}}{2} \]
\[ (8) \quad \frac{3}{2} + j\frac{3\sqrt{3}}{2} \]

*Impedance is defined as:*

\[ Z = \frac{V}{I} = \frac{V_0 e^{j\phi_V}}{I_0 e^{j\phi_I}} = \frac{V_0}{I_0} e^{j(\phi_V - \phi_I)} \]

*From the plot, we know:*

\[ V_0 = 6 \]
\[ I_0 = 2 \]
\[ \phi_V - \phi_I = -60^\circ \]

*Therefore,*

\[ Z = \frac{6}{2} e^{-60^\circ} = 3 e^{-60^\circ} \]

Transforming to cartesian form:

\[ Z = \frac{3}{2} - \frac{\sqrt{3}}{2} j \]

30-60-90 triangle $\Rightarrow x = \frac{\sqrt{3}}{2}$, $y = \frac{3}{2}$

\[ Z = \frac{3}{2} - \frac{\sqrt{3}}{2} \Omega \]
Potentially Useful Formulas

First order circuit: \( x(t) = x(\infty) + \left[ x(t^*) - x(\infty) \right] e^{-t/\tau} \), \( \tau = L/R \) for LR circuit, \( \tau = RC \) for RC circuit

Series RLC: \( s^2 + \frac{R}{L} s + \frac{1}{LC} = 0 \)

Parallel RLC: \( s^2 + \frac{1}{RC} s + \frac{1}{LC} = 0 \)

\[
x(t) = x(\infty) + \left( Ae^{s_1 t} + Be^{s_2 t} \right) \quad \text{--- Over-damped, } b^2 - 4c > 0
\]

\[
x(t) = x(\infty) + \left( A + Bt \right) e^{-\sigma t} \quad \text{--- Critically-damped, } b^2 - 4c = 0
\]

\[
x(t) = x(\infty) + \left( A \cos \omega_d t + B \sin \omega_d t \right) e^{-\sigma t} \quad \text{--- Under-damped, } b^2 - 4c < 0
\]

\[
s_1, s_2 = \frac{-b \pm \sqrt{b^2 - 4c}}{2} \quad \text{for } s^2 + bs + c = 0, \text{ where } c = (LC)^{-1}
\]

\[
\sigma = \frac{b}{2} = \begin{cases} \frac{R}{2L} & \text{(series)} \\ \frac{1}{2RC} & \text{(parallel)} \end{cases}
\]

\[
\omega_o = \frac{1}{\sqrt{LC}}
\]

\[
s_{1,2} = -\sigma \pm \sqrt{\sigma^2 - \omega_o^2}
\]

\[
\omega_d = \frac{\sqrt{4c - b^2}}{2} = \sqrt{\omega_o^2 - \sigma^2}
\]
TABLE 9.2  General Solutions for Constant-Source Second-Order Networks

General solution of the driven differential equation
\[
\frac{d^2x}{dt^2} + b \frac{dx}{dt} + cx = F
\]

having characteristic equation \( s^2 + bs + c = (s - s_1)(s - s_2) = 0 \), with roots
\[
s_1, s_2 = \frac{-b \pm \sqrt{b^2 - 4c}}{2}
\]

Case 1. Real and distinct roots; \( b^2 - 4c > 0 \):
\[
x(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} + X_F
\]

where \( X_F = F/c \), and
\[
x(0^+) = K_1 + K_2 + X_F
\]
\[
x'(0^+) = s_1 K_1 + s_2 K_2
\]

Case 2. The roots \( s_1 = -\sigma + j\omega_d \) and \( s_2 = -\sigma - j\omega_d \) of the characteristic equation are distinct but complex; \( b^2 - 4c < 0 \):
\[
x(t) = e^{-\sigma t} [A \cos(\omega_d t) + B \sin(\omega_d t)] + X_F
\]

where again \( X_F = F/c \), and
\[
x(0^+) = A + X_F
\]
\[
x'(0^+) = -\sigma A + \omega_d B
\]

Case 3. The roots are real and equal; \( s_1 = s_2 \) and \( b^2 - 4c = 0 \):
\[
x(t) = (K_1 + K_2 t)e^{s_1 t} + X_F
\]

where again \( X_F = F/c \), and
\[
x(0^+) = K_1 + X_F
\]
\[
x'(0^+) = s_1 K_1 + K_2
\]