# ME 475 Final Exam
**Tuesday, May 6$^{th}$, 2014**

**Name** Solution

**Instructions**

(1) This is a closed book examination, but you are allowed two 8.5×11 crib sheets.

(2) You have two hours to work all five problems on the exam.

(3) Use the solution procedure we have discussed: what are you given, what are you asked to find, what are your assumptions, what is your solution, does your solution make sense. You must show all of your work to receive any credit.

(4) You must write neatly and should use a logical format to solve the problems. You are encouraged to really "think" about the problems before you start to solve them. Please write your name in the top right-hand corner of each page.

(5) A table of Laplace and Z transform pairs is attached at the end of this exam set.

| | |
|---|---|
| **Problem No. 1 (12 points)** | ________ |
| **Problem No. 2 (15 points)** | ________ |
| **Problem No. 3 (18 points)** | ________ |
| **Problem No. 4 (10 points)** | ________ |
| **Problem No. 5 (25 points)** | ________ |
| **Problem No. 6 (20 points)** | ________ |
| **TOTAL** | ________ |

**<u>PROBLEM NO. 1</u> (12 points)**

Figure 1 shows the Bode plot of four transfer functions:

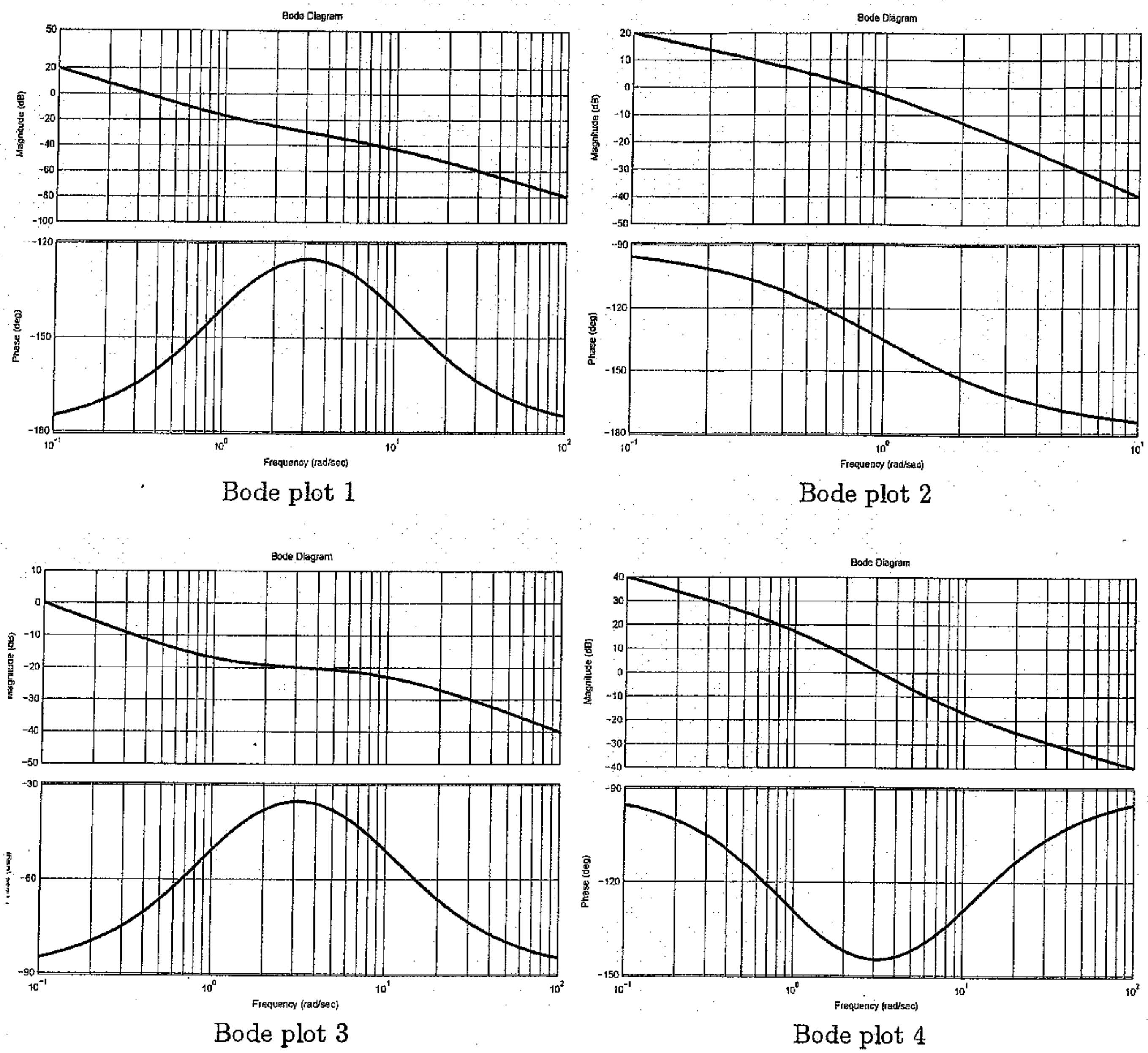


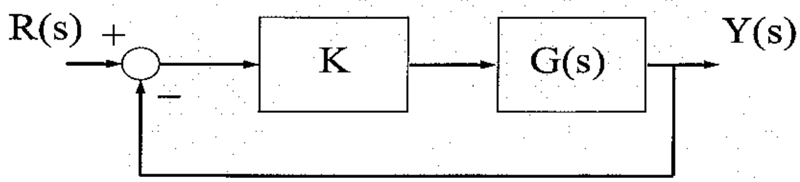

Figure 1. Bode plot of four transfer functions.

In the following table, please match the four Bode plots shown in Figure 1 to their corresponding transfer function by filling the blanks with bode plot numbers 1, 2, 3, 4. <u>You need to justify your answer</u>.

| Transfer function | $\frac{s+1}{s(s+10)}$ | $\frac{1}{s(s+1)}$ | $\frac{s+1}{s^2(s+10)}$ | $\frac{s+10}{s(s+1)}$ |
|---|---|---|---|---|
| Corresponding Bode plot number | 3 | 2 | 1 | 4 |
| | $-90^\circ$ high freq. phase first ~~down~~ up then ~~up~~ down (zero before pole) | $-180^\circ$ phase at high freq. Type 1 system | type 2; begins with $-40$dB/dec; $-180^\circ$ @ high freq. | $-90^\circ$ high freq. phase first down then up (pole before zero) |

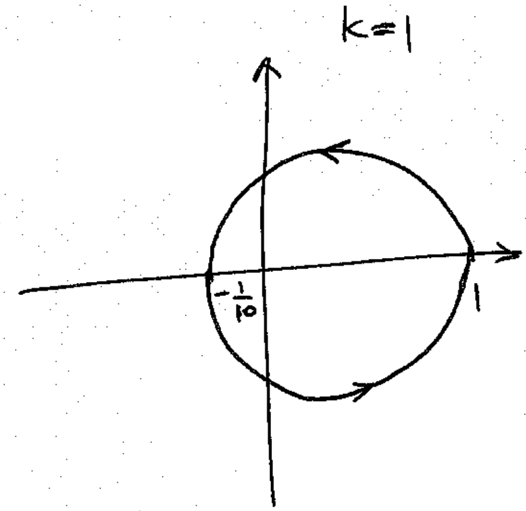
**PROBLEM NO. 2** **(15 points)**

Consider the following feedback system, where $G(s) = \dfrac{s+1}{s-10}$.




(a) (10 points) Let K=1, sketch the Nyquist plot for the (open) loop transfer function. Clearly identify the locations where the Nyquist plot crossed the real axis.

(b) (5 points) Using the Nyquist stability criterion and the Nyquist plot you obtained in (a), determine the range of K that will guarantee closed-loop stability.




(a) $G(j\omega) = \dfrac{j\omega + 1}{j\omega - 10}$

$|G(j\omega)| = \dfrac{\sqrt{1+\omega^2}}{\sqrt{100+\omega^2}}$

$\angle G(j\omega) = \angle(1+j\omega) - \angle(-10+j\omega)$

$K=1$: $\omega = 0$: $G(j\omega) = \frac{-1}{10}$

$\omega \to \infty$: $G(j\omega) \approx 1$

$\omega = 1$: $|G(j\omega)| = 0.14$

$\angle G(j\omega) = 45^\circ - 174.3^\circ = -129.3^\circ$

$\omega = 2$: $|G(j\omega)| = 0.22$

$\angle G(j\omega) = 63.4^\circ - 168.7^\circ = -110.3^\circ$

$\vdots$

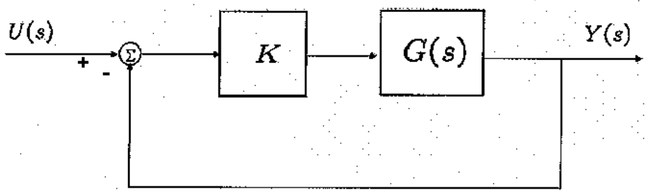
(b) $Z = N + P$ where $P = 1$

want $Z = 0$, i.e. $N = -1$ for stability.

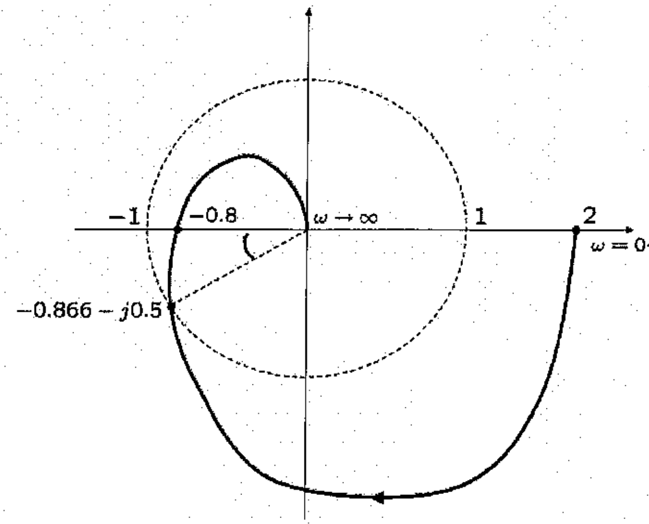
$\therefore$ $K > 10$ for stable system. $(N = -1)$;

**PROBLEM NO. 3** **(15 points)**

In the unity feedback system shown in Figure 2(a), the transfer function G(s) has a Nyquist plot ($0 \le \omega < \infty$) shown in Figure 2(b). Assume that the Nyquist plot $G(j\omega)$ starts at $\omega = 0$ from the point $s = 2$ on the positive real axis, intersects the negative real axis at the point $s = -0.8$, and intersects the unit circle at the point $s = -0.866 - j0.5$.




(a) A feedback system




(b) Nyquist plot of G(jω).

Figure 2. Problem 3.

(a) (5 points) What are the gain margin (GM) and the phase margin (PM) of G(s)?

(b) (5 points) Suppose we know that G(s) is stable. What is the range of K > 0 so that the closed-loop system in Figure 6 (a) is stable?

(c) (5 points) What is the system type of G(s), i.e., the number of its poles at the origin? For the system in Figure 6 (a), suppose K = 1. What is its steady state error in tracking the unit step input? What is its steady state error in tracking the unit ramp input?

(a) $GM = \frac{1}{0.8} = 1.25$

$PM = 180^\circ + \angle(-0.866 - j0.5) = 180^\circ - 150^\circ = 30^\circ$

(b) $0 < K < GM$ since $Z = N + P$ where $P = 0$, therefore, in order to have $Z = 0$, we want $N = 0$, no encirclement of $(-1, j0)$ point.

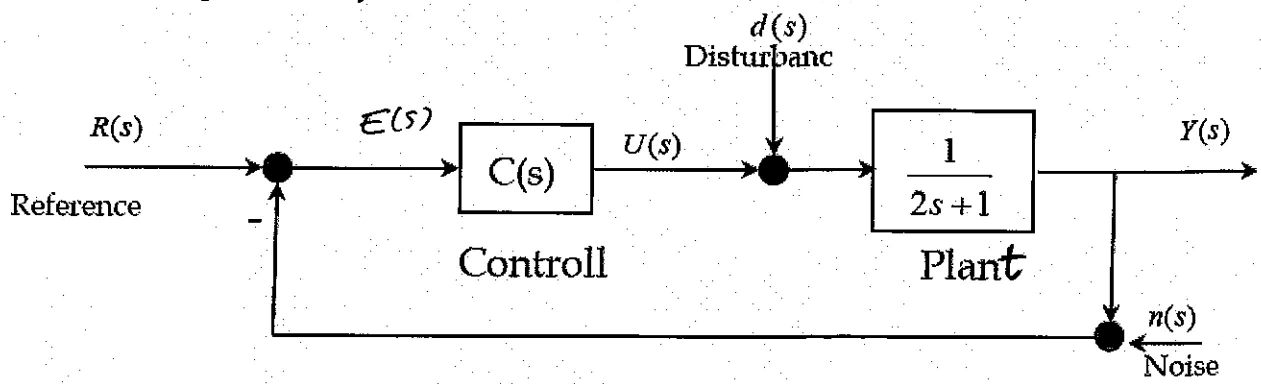
(c) System is of type 0 since $G(j\omega)$ starts from $G(0) = 2$;

$e_{ss\_step} = \frac{1}{1+G(0)} = \frac{1}{1+2} = \frac{1}{3}$

$e_{ss\_ramp} = \infty$.

**PROBLEM NO. 4 (14 points)**

Consider the following feedback system:




The goal is to design a controller C(s) to meet the following performance specifications:

(i). The steady-state error for sinusoidal disturbances with $\omega < 0.1$ rad/sec and unity amplitude (i.e., $d(t) = \sin(\omega t)$ and $\omega < 0.1$ is less than 1/500.
(ii). Measurement noises n(s) at frequency greater than 100rad/sec are to be attenuated at the output by at least a factor of 100.

To achieve the above goals, it is necessary to transform the above requirements into some design constraints in terms of the frequency response of the open-loop transfer function L(s)=C(s)P(s),

(a) (7 points) Determine the constraint that L(s) has to satisfy to meet the performance specification (i).
(b) (7 points) Determine the constraint that L(s) has to satisfy to meet the performance specification (ii).

(a) $\frac{E(s)}{D(s)} = \frac{-\frac{1}{2s+1}}{1+L(s)}$, want $\left|\frac{E(s)}{D(s)}\right| < \frac{1}{500}$ $\forall\ \omega < 0.1$

$$\frac{|G(s)|}{|1+L(s)|} = \frac{|G(j\omega)|}{|1+L(j\omega)|} = \frac{\frac{1}{\sqrt{4\omega^2+1}}}{|1+L(j\omega)|} < \frac{1}{500} \quad \forall\ \omega < 0.1$$

$$|1+L(j\omega)| > \frac{500}{\sqrt{4\omega^2+1}} = 500 \quad \forall\ \omega < 0.1$$

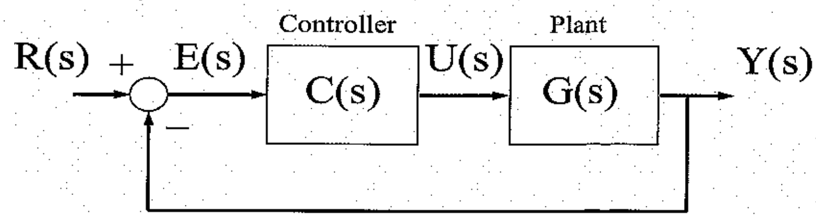
i.e. $|L(j\omega)| > 500$

(b) $\frac{Y(s)}{N(s)} = \frac{-L(s)}{1+L(s)}$ $\quad \left|\frac{Y(j\omega)}{N(j\omega)}\right| = \frac{|L(j\omega)|}{|1+L(j\omega)|} < \frac{1}{100}$ $\forall \omega > 100$

want $L(j\omega)$ ~~large~~ small, $\forall\ \omega > 100$, $\frac{|L(j\omega)|}{|1+L(j\omega)|} \approx |L(j\omega)| < \frac{1}{100}$

**PROBLEM NO. 5 (25 points)**

Consider the following feedback system:

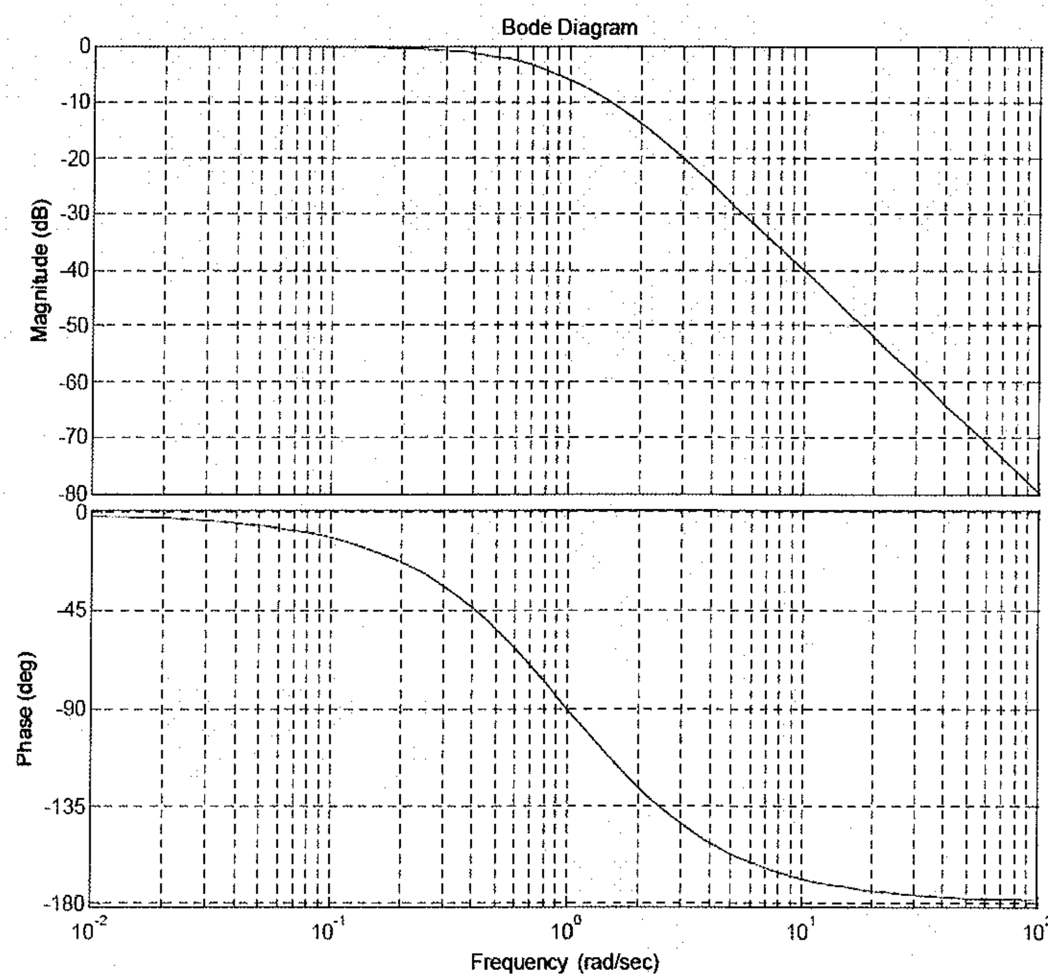



The Bode plot of the plant G(s) are shown in Figure 5. You are required to design a controller C(s) to meet the following performance specifications:

(P1). Steady-state error to a unit step reference input should be 0.001.
(P2). Gain crossover frequency should be 10 rad/sec to have a reasonable fast response.
(P3). Phase margin should be at least $55^0$ to avoid large OS%.

In the figure below, the magnitude and phase of G(s) at 10 rad/sec are 0.01 (or –40dB) and $-169^0$ respectively. If you need other magnitude or phase information, you can estimate them from the figure below.

Design a lead-lag compensator to achieve all the above performance specifications (P1-P3). If you need to include some additional phase angle, use $\phi_e = 10^0$.



**PROBLEM NO. 5 (continued)**

$$\frac{1}{1+K_{p\_desired}} = 0.001$$

$$K_{p\_plant} = 1$$

$$K_{p\_desired} = 999 \qquad\qquad K_{p\_desired} = 999$$

Lead: $\omega_{gc} = 10$ rad/s

$$\phi_m = 55^\circ + 10^\circ - (-180^\circ + 169^\circ) = 55^\circ - 11^\circ + 10^\circ = 54^\circ$$

$$\alpha = \frac{1+\sin\phi_m}{1-\sin\phi_m} = 9.47 \qquad \sqrt{\alpha} = 3.077$$

$$\omega_1 = \frac{10}{\sqrt{\alpha}} = 3.25 \qquad \omega_2 = \omega_1 \alpha = 30.78$$

$$KG_{Lead}(s) = 999 \cdot \frac{1+ s/3.25}{1+ s/30.78} = 999 \cdot \frac{1+0.308s}{1+0.0325s}$$

Lag: Need to compensate for

$$\underbrace{-20\lg\sqrt{\alpha}}_{\text{Lead}} + \underbrace{20\lg K}_{\text{gain}} + \underbrace{(-40\,dB)}_{\text{plant}}$$

$$= 9.76 + 60 + (-40) = 29.76\ dB = 20\lg\beta$$

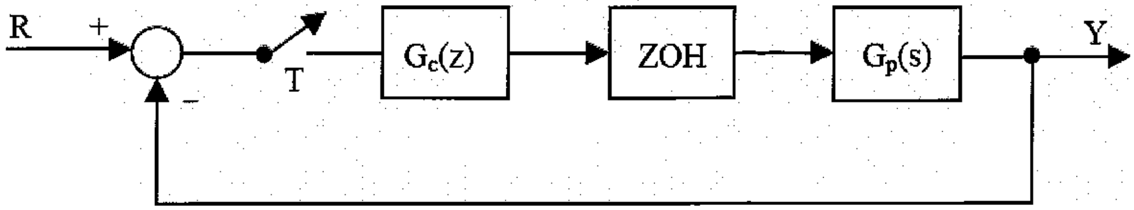
$$-20\lg\beta = -29.76 \Rightarrow \beta = 30.76$$

$$\omega_z = 1 \qquad \omega_p = \frac{\omega_z}{\beta} = 0.0325$$

$$C(s) = 999 \cdot \frac{1+0.308s}{1+0.325s} \cdot \frac{1+s}{1+30.77s}$$

**PROBLEM NO. 6 (20 points)**

Consider the following feedback system:




where $G_C(z) = K$ and $G_P(z) = \frac{1}{(s+5)}$.

Assuming a sampling period of T. Find:

(a) (10 points) The discrete transfer functions (Z-transfer functions) of the open-loop and closed-loop systems.

(b) (5 points) The Z-transform of the output Y(z) for a unit step input R(z) and use the final value theorem to find the steady state response.

(c) (5 points) If $K=10$, for what values of $T$ is the closed-loop system stable?

(a) $ZOH = \frac{1-e^{-Ts}}{s}$

$$G(z) = K \cdot \mathcal{Z}\left[\left(\frac{1-e^{-Ts}}{s}\right)\cdot\frac{1}{s+5}\right]$$

$$= K\cdot(1-z^{-1})\,\mathcal{Z}\left\{\frac{1}{s(s+5)}\right\}$$

$$= K\cdot(1-z^{-1})\cdot\mathcal{Z}\left\{\frac{A}{s}+\frac{B}{s+5}\right\}$$

$As+5A+Bs=1$

$\left.\begin{array}{l} A+B=0 \\ 5A=1 \end{array}\right\} \Rightarrow A=0.2,\ B=-0.2$

$$= K\cdot(1-z^{-1})\cdot\mathcal{Z}\left\{\frac{0.2}{s}-\frac{0.2}{s+5}\right\}$$

$$= K\cdot(1-z^{-1})\left[0.2\cdot\frac{z}{z-1} - 0.2\cdot\frac{z}{z-e^{-5T}}\right]$$

$$= K\cdot\frac{z-1}{z}\cdot 0.2\cdot\left[\frac{z}{z-1}-\frac{z}{z-e^{-5T}}\right]$$

$$= 0.2K\cdot\left[1-\frac{z-1}{z-e^{-5T}}\right]$$

$$= 0.2K\cdot\frac{1-e^{-5T}}{z-e^{-5T}}$$

**PROBLEM NO. 6 (continued)**

$$T(z) = \frac{G(z)}{1+G(z)} = \frac{0.2K \cdot \dfrac{1-e^{-5T}}{z-e^{-5T}}}{1+0.2K\,\dfrac{1-e^{-5T}}{z-e^{-5T}}}$$

$$= \frac{0.2K(1-e^{5T})}{z+0.2K-(1+0.2K)e^{-5T}}$$

(b) $$\lim_{z\to 1} Y(z)(z-1) = \frac{0.2(1-e^{-5T})\cdot K}{1+0.2K-(1+0.2K)e^{-5T}} \qquad \left(U(z)=\frac{z}{z-1}\right)$$

(c) $$z = -0.2K + (1+0.2K)\,e^{-5T}$$

$K=10$:

$$z = -2+3e^{-5T}$$

want $|z|<1$ : $\quad -1 < -2+3e^{-5T} < 1$

i.e. $\quad 3e^{-5T} > 1 \;\;\&\;\; 3e^{-5T} < 3$

$$T < 0.22 \;\;\&\;\; T > 0.$$

$$0 < T < 0.22$$