MATH 265 Second Midterm Examination, November 13, 2007

Name : ____________________
PUID : ____________________

No books, notes, or calculators allowed. Please be neat and show all your work on these pages (use back of pages if necessary). Return all these pages. Work must be shown for full credit.

Problem Number	Possible Points	Points Earned
1	20	
2	20	
3	20	
4	20	
5	20	
TOTAL	100	

6

1. The plane **W** has an orthogonal basis $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. Let $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

a) Find the projection of **b** onto **W**, $\text{proj}_W(\mathbf{b})$. (8 points)

Since $u_1 \cdot u_2 = 0$

$$\text{proj}_W(b) = \frac{(b, u_1)}{(u_1, u_1)} \vec{u}_1 + \frac{(b, u_2)}{(u_2, u_2)} \vec{u}_2$$

$$= \frac{4}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{0}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

b) Find the distance from **b** to **W**. (7 points)

$$\|\vec{b} - \text{proj}_W \vec{b}\| = \left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\|$$

$$= \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$$

c) Find a basis for the orthogonal complement of **W**, $\mathbf{W}^{\perp}$.

Since $\dim W = 2$, $\dim W^{\perp} = 1$

$\vec{b} - \text{proj}_W(b)$ is $\perp$ to W, so $\vec{b} - \text{proj}_W b \in W^{\perp}$

So it is a basis

<u>i.e</u> $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ is a basis for $W^{\perp}$

2. The 4×4 matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 3 & 3 \\ 5 & 5 & 5 & 5 \end{bmatrix}$$

has a row echelon form of

$$B = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

a) (10 points) Find a basis for the subvector space $\text{Row}(\mathbf{A}) \subset \mathbb{R}_4$. Justify your answer.

row(A) = row(B), row(B) has basis
[1 1 2 2], [0 0 1 1], so this is a basis of row A

or dim Row B = dim Row A = 2
so any 2 indep rows of A are a basis,
eg [1 1 2 2], [2 2 1 1]
etc.

b) (10 points) Find a basis for the subvector space $\text{Col}(\mathbf{A}) \subset \mathbb{R}^4$. Justify your answer.

leading 1s in row ech. form B are in
cols 1 & 3, so col. 1 & 3 of A are a basis,

$$\begin{bmatrix} 1 \\ 2 \\ 2 \\ 5 \end{bmatrix} \& \begin{bmatrix} 2 \\ 1 \\ 3 \\ 5 \end{bmatrix}$$

or do col. ops to A (not B!).

3. The matrix **A** from problem 2 above has reduced row echelon form

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

a) (12 points) Give a basis for the null space, Null(**A**), the set of solutions to the homogeneous equation $A\mathrm{x} = 0$.

from C, $x_1 + x_2 = 0$, $x_3 + x_4 = 0$, so

$x_1 = -x_2$ & $x_3 = -x_4$

i.e $\mathbb{X} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

so $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ are a basis.

b)(8 points) Give a basis for the orthogonal complement $\mathrm{Row}(\mathbf{A})^\perp$, with respect to the usual dot product in $\mathbb{R}_4$. Explain your answer.

$\mathrm{Row}(A)^\perp = \mathrm{Null}(A)^T$

so basis for $\mathrm{Row}(A)^\perp$ is $[-1\ 1\ 0\ 0]$ &
$[0\ 0\ -1\ 1]$

or notice $[1\ 1\ 0\ 0]$, $[0\ 0\ 1\ 1]$ is orthogonal basis of row(A),
and extend by Gram Schmidt?

4. Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 5 \end{bmatrix}.$$

Notice that $\mathbf{b}$ is not in the column space of $\mathbf{A}$, so there is no exact solution to $A\mathbf{x} = \mathbf{b}$.

a) Find the least squares solution $\mathbf{y}$ to $A\mathbf{y} = \mathbf{b}$. Hint: multiply on the left by A^T, then solve for $\mathbf{y}$. (15 points)

$$A^TAy = A^Tb$$

$$\begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} y = \begin{bmatrix} 6 & 1 \\ 1 & 3 \end{bmatrix} y = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 2 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 6+1-5 \\ 3-2+5 \end{bmatrix} = \begin{matrix} 2 \\ 6 \end{matrix}$$

$$\left[\begin{array}{cc|c} 6 & 1 & 2 \\ 1 & 3 & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 6 \\ 6 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 6 \\ 0 & -17 & 2-36 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 6 \\ 0 & 1 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right] \quad \begin{matrix} x_1 = 0 \\ x_2 = 2 \end{matrix} \quad \to y = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

b) Find the error, $||A\mathbf{y} - \mathbf{b}||$.(5 points).

$$\text{error} = ||Ay - b|| = \left|\left| \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -4 \\ -3 \end{bmatrix} \right|\right|$$

$$= \sqrt{1+1+16+9} = \sqrt{27} = 3\sqrt{3}$$

5. a) (10 points) Prove or disprove: $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ is in the span of $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\2\\3\\4\end{bmatrix}$, and $\begin{bmatrix}2\\1\\1\\1\end{bmatrix}$.

$$\left[\begin{array}{ccc|c}1&0&2&1\\0&2&1&2\\1&3&1&3\\0&4&1&4\end{array}\right] \to \left[\begin{array}{ccc|c}1&0&2&1\\0&2&1&2\\0&3&-1&2\\0&4&1&4\end{array}\right] \to \left[\begin{array}{ccc|c}1&0&2&1\\0&-1&2&0\\0&3&-1&2\\0&4&1&4\end{array}\right]$$

$$\to \begin{array}{cccc}1&0&2&1\\0&1&-2&0\\0&0&5&2\\0&0&9&4\end{array} \to \left.\begin{array}{ccc|c}1&0&2&1\\0&1&-2&0\\0&0&1&5/2\\0&0&1&4/9\end{array}\right.$$

inconsistent so not in span

b) (10 points) Prove or disprove: the vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\begin{bmatrix}1\\1\\1\end{bmatrix}$, and $\begin{bmatrix}1\\2\\1\end{bmatrix}$ are linearly independent in $\mathbb{R}^3$.

$$\begin{bmatrix}1&1&1\\0&1&2\\1&1&1\end{bmatrix} \to \begin{array}{ccc}1&1&1\\0&1&2\\0&0&0\end{array} \to \begin{array}{ccc}1&0&-1\\0&1&2\\0&0&0\end{array}$$

So $v_3 = 2v_2 - v_1 = 2\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}2-1\\2-0\\2-1\end{bmatrix} = \begin{bmatrix}1\\2\\1\end{bmatrix}$

So not independent