Math 265 Linear Algebra **Final Exam** **Spring 2001**

Student Name (print):	Student ID:

Circle the name of your instructor (with the time of your class):

Ban   De la Cruz (9:00)   De la Cruz (10:30)   Corless   Feng

Gottlieb (9:00)   Gottlieb (1:30)   Krushchev (10:30)   Krushchev (11:30)

Matsuki (1:30)   Matsuki (2:30)   Pascovici

Walther   Włodarczyk (9:00)   Włodarczyk (10:30)

Do not write below this line.

Please be neat and show all work.
Write each answer in the provided box.
Use the back of the sheets and the last 3 pages for extra scratch space.
Return this entire booklet to your instructor.
No books. No notes. No calculators.

Problem #	Max pts.	Earned points
1	20	
2	8	
3	8	
4	8	
5	8	
6	8	
7	8	
8	8	
9	8	
10	8	
11	8	
Section I	**100**	

12	10	
13	10	
14	10	
Section II	**30**	
15	20	
16	15	
17	15	
18	20	
Section III	**70**	
TOTAL	**200**	

Section I: Short problems

No partial credit on this part, but show all your work anyway. It might help you if you come close to a borderline. Please be neat. Write your answer in the provided box.

1. It is given that $A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$, $\mathbf{rref}(A) = \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find the rank of A.

(b) Find a basis for the null space of A.

(c) Find a basis for the column space of A. We require that you choose the vectors for the basis from the column vectors of A.

(d) Find a basis for the row space of A. We require that you choose the vectors for the basis from the row vectors of A.

2. Determine the value(s) of a so that the following linear system has no solution.

$$\begin{cases} x_1 & + 2x_2 & + x_3 & = & a \\ x_1 & + x_2 & + ax_3 & = & 1 \\ 3x_1 & + 4x_2 & + (a^2-2)x_3 & = & 1. \end{cases}$$

3. Find the standard matrix for the linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$ such that

$$L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}, L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}, L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\-2\end{bmatrix}.$$

4. Determine the value(s) of a so that the line whose parametric equations are given by

$$\begin{cases} x = -3 + t \\ y = 2 - t \\ z = 1 + at \end{cases}$$

is parallel to the plane

$$3x - 5y + z + 3 = 0.$$

5. Find the symmetric equations of the line which is the intersection of the following two planes: $x + y - z = 2, 3x + 4y + z = 5$.

6. Compute the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

7. E is a 3×3 matrix of the form

$$E = \begin{bmatrix} 1 & 8 & 3 \\ x & y & z \\ -3 & 7 & 2 \end{bmatrix}.$$

Given $\det(E) = 5$, compute the determinant of the following matrix

$$F = \begin{bmatrix} x & y & z \\ 1 & 8 & 3 \\ -3+4x & 7+4y & 2+4z \end{bmatrix}$$

8. Find the matrix G such that

$$adj(G) = \begin{bmatrix} 2 & 4 \\ -5 & 7 \end{bmatrix}.$$

9. Find the dimension of the subspace $V = span\{v_1, v_2, v_3, v_4\}$ in $\mathbb{R}^3$ where

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

10. Find the projection $\text{Proj}_W v$ of the vector $v = \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}$ onto the subspace W spanned by

$$\left\{ v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

11. We have a subspace W in $\mathbb{R}^4$ spanned by the following three linearly independent vectors

$$\left\{ u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 3 \\ -3 \\ 0 \\ -2 \end{bmatrix} \right\}.$$

Find an orthonormal basis of W.

Section II: Multiple choice problems

For Problems 12 through 15, circle only one (the correct) answer for each part. No partial credit.

12. Let A be a 3×3 matrix with $\det(A) = 0$. Determine if each of the following statements is true or false.

(a) $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. True False

(b) $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$. True False

(c) For every 3×3 matrix B, we have $\det(A + B) = \det(B)$. True False

(d) For every 3×3 matrix B, we have $\det(AB) = 0$. True False

(e) There is a vector $\mathbf{b}$ in $\mathbb{R}^3$ such that $rank([A\ \mathbf{b}]) > rank(A)$. True False

13. For each of the following sets, determine if it is a vector (sub)space:

(a) The set of all vectors (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ with the property $2x_1 - x_2 = 0, 3x_3 - x_4 = 0$;

Yes No

(b) The set of all vectors (x_1, x_2, x_3) in $\mathbb{R}^3$ with the property $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$;

Yes No

(c) The set of all vectors (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ with the property $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$;

Yes No

(d) The set of all vectors of the form $(a + b - 1, 2a + 3c - 1, b - c, a + b + c + 2)$ in $\mathbb{R}^4$ where a, b and c are arbitrary real numbers;

Yes No

(e) The set of all solutions to the linear system of differential equations $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ where $A = \begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix}$;

Yes No

14. For the problems (a), (b) and (c), determine if the given set of vectors is linearly independent or linearly dependent:

(a) $\left\{ \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$; Independent Dependent

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$; Independent Dependent

(c) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix} \right\}$; Independent Dependent

For the problems (d) and (e), determine if the given set of vectors spans $\mathbb{R}^3$:

(d) $\left\{ \begin{bmatrix} \pi \\ 2\pi \\ -1\pi \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$; span not span

(e) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$; span not span

Section III: Multi-Step problems

Show all work (no work - no credit!) and display computing steps. Write clearly.

15. Let

$$A = \begin{bmatrix} -15 & 28 \\ -8 & 15 \end{bmatrix}.$$

(a) Find the eigenvalues and compute an eigenvector for each eigenvalue.

(b) Find an invertible matrix P and a diagonal matrix D such that

$$P^{-1}AP = D.$$

(c) Compute A^{37}.

16. Find the least squares fit line for the points

$$(-2,1),(-1,3),(0,2),(1,3),(2,1).$$

17. Let

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \\ \dfrac{dx_2}{dt} \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

be the linear system of differential equations where

$$A = \begin{bmatrix} 3 & -5 \\ 5 & 3 \end{bmatrix}.$$

(a) Find the eigenvalues and find an eigenvector for each eigenvalue for A.
Note: The eigenvalues are COMPLEX-valued.

(b) Find the general REAL solution to the linear system of differential equations.

18. Let

$$\begin{cases} \dfrac{dx_1}{dt} = 2x_1 + 5x_2 \\[2ex] \dfrac{dx_2}{dt} = 3x_1 + x_2 + 3x_3 \\[2ex] \dfrac{dx_3}{dt} = -x_1 \end{cases}$$

be a linear system of differential equations.

(a) Find the eigenvalues and find an eigenvector for each eigenvalue for the coefficient matrix of the linear system of differential equations.

(b) Find the general solution to the linear system of differential equations.

(c) Find the solution to the initial value problem

$$x_1(0) = 4, x_2(0) = 16, x_3(0) = 0.$$