Practice Midterm 2

Everywhere in what follows the vector spaces involved are finite-dimensional. All bases appearing will be ordered bases.

I. Choose the unique correct answer:

(1) Let $T : P_2(\mathbb{R}) \to M_2(\mathbb{R})$, $T(f) \overset{\text{def}}{=} \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$.

Let $\alpha = \{1, x, x^2\}$ and $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

Then $[T]_\alpha^\beta$ is:

a) $\begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}$ c) $\begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}$

d) $\begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(2) Let's consider $U, T : V \to W$ to be linear transformations and the following equalities:

$$(U+T)(x+y) = U(x+y) + T(x+y) = U(x) + U(y) + T(x) + T(y)$$
$$= [U(x) + T(x)] + [U(y) + T(y)] = (U+T)(x) + (U+T)(y).$$

These equalities prove that:

a) $U+T$ is a linear transformation

b) $U+T$ is injective (one-to-one)

c) $U+T$ behaves well w.r.t. addition

d) $U+T$ is invertible

3. Let $U, T : V \longrightarrow W$ be linear transformations ($\dim V = n$, $\dim W = m$) and α, β fixed bases in V, W respectively. Then the equality

$$[T+U]_\alpha^\beta = [T]_\alpha^\beta + [U]_\alpha^\beta$$

represents:

a) The good behavior of $U+T$ w.r.t. addition

b) The linearity of $U+T$

c) The good behavior of f w.r.t. addition, where f is the isomorphism between $\mathcal{L}(V, W)$ and $\mathcal{M}_{m\times n}(\mathbb{R})$.

d) The linearity of the above f

4) Let $T : V \to W$ be a linear transformation and let's consider the assertions :

1) T invertible iff T bijective

2) T invertible $\Rightarrow T^{-1}$ invertible

3) T invertible $\Rightarrow \operatorname{rank}(T) = \operatorname{rank}(T^{-1})$

4) T invertible $\Rightarrow T^{-1}(ax) = a\,T^{-1}(x)$ all $a \in \mathbb{R}$ and all $x \in W$

Then :

a) All assertions are true

b) Only 3) is not true

c) Only 1) and 2) are true

d) Only 2) is not true

5) Let T be an isomorphism between V and W.

Consider the assertions :

1) If $\{\vec{v_1}, \dots, \vec{v_n}\}$ is a basis for V, then $\{T(\vec{v_1}), \dots, T(\vec{v_n})\}$ is a basis for W

2) $\dim(V) = \dim(W)$

3) If $\{\vec{v_1}, \dots, \vec{v_n}\}$ is a basis for V, then $W \simeq \mathbb{R}^n$.

4) If $\{\vec{v_1}, \dots, \vec{v_n}\}$ is linearly independent in V, then $\{T(\vec{v_1}), \dots, T(\vec{v_n})\}$ is linearly independent in W

5) If $\{\vec{v_1}, \dots, \vec{v_n}\}$ generates V, then $\{T(\vec{v_1}), \dots, T(\vec{v_n})\}$ generates W.

Then :

a) All are true

b) Only 3) is not true

c) Only 1) and 2) are true

d) Only 5) is not true

6 Let $T : V \to W$ be a linear transformation and α, β bases in V, W respectively. Let's consider the assertions :

1) T bijective $\Rightarrow [T]_\alpha^\beta$ invertible

2) $[T]_\alpha^\beta$ invertible $\Rightarrow \text{rank}(T) = \dim(V)$

3) $[T]_\alpha^\beta$ invertible $\Rightarrow \text{rank}(T) = \dim(W)$

4) $\overbrace{[T^{-1}T]_\alpha^\alpha = I_n}^{\text{(if } T \text{ invertible and } \dim V = n\text{)}}$; 5) $[1_W]_\beta^\beta = [1_V]_\alpha^\alpha$

Then :

a) All are true

b) Only 5) is not true

c) Only 3) is not true

d) Only 4) is not true

7) Let $\alpha = \{1, x, x^2\}$ and $\beta = \{2x^2 - x, 3x^2 + 1, x^2\}$ be two bases of $P_2(\mathbb{R})$. Then the change of coordinates matrix from the basis α to the basis β is :

a) $\begin{pmatrix} 0 & 1 & -3 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 0 & 1 \\ -3 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$

d) $\begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix}$

8) Let $T : P_1(\mathbb{R}) \to P_1(\mathbb{R})$, $T(f) = f(-1) + f(1)\cdot(1+x)$.
Then the eigenvalues of T are :

a) 1 and 2 b) 2 and 3 c) 1 and 3 d) 1 and 4

9) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be an invertible linear operator and α, β two bases for $\mathbb{R}^2$. Let $\vec{v}$ be a vector in $\mathbb{R}^2$.

If $[\vec{v}]_\beta = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $[T]_\alpha^\beta = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$, find $[T^{-1}(\vec{v})]_\alpha$.

a) $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ b) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ c) $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ d) $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

10) In the proof of the theorem stating that the solution set of the homogeneous system $Ax = 0$ is the kernel of L_A and its dimension equals $n - \text{rank}(A)$, how many of the following are used?

1) The definition of the kernel of a linear transformation
2) The definition of the left-multiplication transformation L_A
3) The dimension theorem

a) All of them b) All but 1) c) All but 2) d) All but 3)

11) If $A = \begin{pmatrix} -2 & 0 \\ 3 & 1 \end{pmatrix}$, find A^n, where n is an integer ≥ 1.

a) $\begin{pmatrix} (-2)^n & 0 \\ -(-2)^n+1 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 2^n & 0 \\ 2^n+1 & 1 \end{pmatrix}$ c) $\begin{pmatrix} (-2)^n & 0 \\ -(-2)^n+1 & 0 \end{pmatrix}$ d) $\begin{pmatrix} 2^n & 0 \\ -(-2)^n+1 & 2 \end{pmatrix}$

II. For each of the following assertions specify if it is true or false.

(12) Any two finite-dimensional vector spaces having the same dimension are isomorphic to one another.

a) True b) False

(13) A square matrix $A \in \mathcal{M}_n(\mathbb{R})$ is invertible iff there exists $A^{-1} \in \mathcal{M}_n(\mathbb{R})$ s.t. $A \cdot A^{-1} = I_n$.

a) True b) False

(14) If A, B are square matrices in $\mathcal{M}_n(\mathbb{R})$ s.t. $A \cdot B$ is invertible, then A is invertible.

a) True b) False

(15) Let $T : P_1(\mathbb{R}) \to P_1(\mathbb{R})$, $T(f) = f(-1) + f(1) \cdot (1+x)$.
Then $\{x, 1-x\}$ represents a set of independent eigenvectors, for T.

a) True b) False

16 The above operator T is diagonalizable.

a) True b) False

17 Let $T: V \to V$ be a linear operator and α a basis for V ($\dim(V) = n$). Then 2 represents an eigenvalue of T corresponding to $\vec{v} \in V$ iff 2 represents an eigenvalue of $[T]_\alpha$ corresponding to $[\vec{v}]_\alpha \in \mathbb{R}^n$.

a) True b) False

18 The solution set of any linear system with m equations and n unknowns is a subspace in $\mathbb{R}^n$.

a) True b) False

19 If an invertible matrix A can be obtained from the identity matrix by a finite # of elementary operations, then it can also be obtained from its inverse A^{-1} by a finite # of elementary operations.

a) True b) False

20 By multiplying a matrix A with an invertible matrix at left and with an elementary matrix at right we do not change the rank of A.

a) True b) False

21) In the proof of the theorem stating that the rank of a matrix stays unchanged through the right-multiplication with an invertible matrix we make use of the definition of a bijective application.

a) True b) False

22) In the proof of the corollary stating that dim (Row space A) = dim (Column space A), we make use of the fact that rank(A) = rank(A^t).

a) True b) False

23) The rank of a matrix A is the maximum (the biggest) of the following two: the maximum # of independent columns of A and the maximum # of independent rows of A.

a) True b) False

24) The following linear system has at least a solution:

$$\begin{cases} x_1 + x_2 - x_3 = 1 \\ 2x_1 + x_2 + 3x_3 = 2 \end{cases}$$

a) True b) False

25 A system of linear equations has at least a solution iff its coefficient matrix A is invertible.

a) True b) False

26 Let $Ax = b$ be a linear system of m equations in n unknowns. If the matrix $(A|b)$ has strictly less than m independent rows, then the system is inconsistent.

a) True b) False

27 If $T: V \to V$ is a linear operator and V does not have a basis consisting in eigenvectors of T, then T is not diagonalizable.

a) True b) False

28 Let $T: V \to V$ be a linear operator, where $\dim(V) = 5$. Suppose T has three real eigenvalues λ_1, λ_2 and λ_3 and λ_2 and λ_3 have order of multiplicity one. Suppose also that there exist three linearly independent eigenvectors corresponding to

the eigenvalue λ_1. Then T is diagonalizable.

a) True b) False

29 The matrix $A = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ is diagonalizable.

a) True b) False

30 Let $T: V \to V$ be a linear operator and α a basis for V ($\dim V = n$). Then $\{\vec{v_1}, \dots, \vec{v_n}\}$ is a basis for V consisting in eigenvectors of T iff $\{[\vec{v_1}]_\alpha, \dots, [\vec{v_n}]_\alpha\}$ is a basis for $\mathbb{R}^n$.

a) True b) False

31 If $T: V \to V$ is a linear operator and $\vec{v}$ is an eigenvector for T corresponding to the eigenvalue λ, then $\vec{v}$ is an eigenvector for T^2 (T composed with T) corresponding to the eigenvalue λ^2.

a) True b) False

32 Let $T: P_1(\mathbb{R}) \to P_1(\mathbb{R})$, $T(f) = f + f(1)\cdot(1+x)$.

Then $\{1+x, 1-x\}$ represents a basis for $P_1(\mathbb{R})$ consisting in eigenvectors of T.

a) True b) False

33 The above operator T is diagonalizable.

a) True b) False