ECE 695C, Midterm #1

6:30-7:30pm Wednesday, February 22, 2012, EE 222,

1. Enter your name, student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. Some time-consuming and bonus questions are assigned a small number of points. Make sure you manage your time well during the exam.
4. There are 4 questions and 10 pages in the exam booklet. Use the back of each page for rough work.
5. You can use a simple calculator. No help sheet is allowed.

Name:

Student ID:

E-mail:

Signature:

Question 1: [25%] Consider the following transmission scheme:
Encoder: We use a binary linear code with the parity-check matrix being

$$H = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}. \tag{1}$$

Channel Model: We assume the following binary Markov channel:

Consider a sequence of Markov random variables $N_1, N_2, \cdots$. For the very first time instant ($t = 1$), we have $P(N_1 = 1) = 0.2$ and $P(N_1 = 0) = 0.8$, and

$$P(N_t = 1|N_{t-1} = 0) = 0.1, \quad P(N_t = 0|N_{t-1} = 0) = 0.9, \tag{2}$$

$$\text{and } P(N_t = 1|N_{t-1} = 1) = 0.8, \quad P(N_t = 0|N_{t-1} = 1) = 0.2 \tag{3}$$

for all $t = 2, 3, 4, \cdots$.

For any input bit X_t at time t, the output bit is $Y_t = X_t \oplus N_t$, the binary exclusive OR between X_t and N_t.

(This Markov channel model is widely used to model the "burst error".)

Question 1 (20%): If we observe $\vec{y}_{\text{obs}} = 100$, what is the most likely (maximum likelihood) codeword?

Question 2 (5%): What is the frame error rate of the maximum likelihood (ML) decoder?

Hint: You may want to work on other questions first since this sub-question may take some time to finish.

Question 2: [20%]

Consider the following binary-input binary-output channel: For any n-bit input string, let n_0 denote the number of 0s in this string and let n_1 denote the number of 1s in this string. (Obviously $n_0 + n_1 = n$.) If we pass this bit string through the given channel, then at most $n_1 \cdot p$ number of 1s will be corrupted and changed to 0. On the other hand, all 0 bits will remain intact. (You can view this channel as an approximation of the Z-channel mentioned in class.)

Suppose you are asked to design a code satisfying the following two conditions:

1. Each codeword has exactly $n \cdot \theta$ 1s and $n \cdot (1 - \theta)$ 0s.

2. There is no decoding error, i.e., no two codewords can lead to the same corrupted observation $\overrightarrow{y}_{\text{obs}}$.

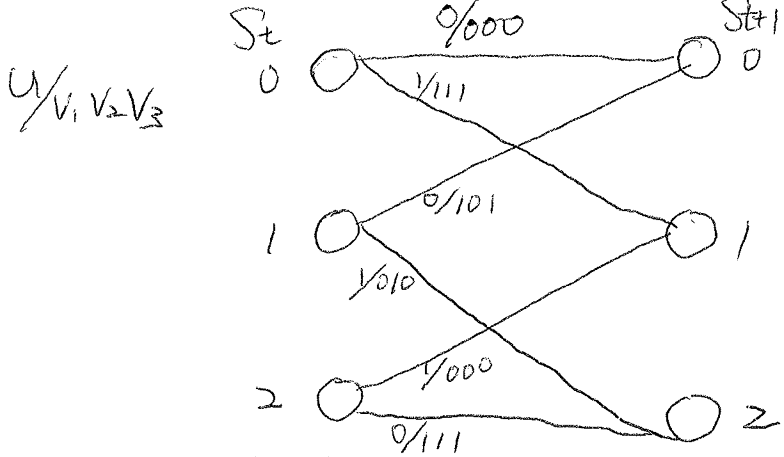
The rate of the code $R(n)$ is denoted by $R(n) \triangleq \frac{\log_2(\text{total number of codewords})}{n}$.

Question: Find the (tightest) sphere-packing bound to upper bound R_∞, which is defined by

$$R_\infty \triangleq \lim_{n \to \infty} R(n). \tag{4}$$

Hint: You may need to use the Stirling's formula $n! \approx n^n$. (The exact Stirling's formula is $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$. However the above simpler approximation is sufficient for this question.)

Question 3: [30%] Consider the following trellis code structure:



In the beginning ($t = 0$), the initial state is $S = 0$.

Question 1 (10%): What is the rate of the code?
Question 2 (10%): If the input bit string is $u_1u_2u_3$ =101, what is the output bit string?
Question 3 (10%): If we pass this code through a binary symmetric channel with crossover probability $p = 0.1$ and we observe $y = 101101110$. What is the most likely value of the second input bit u_2?

Question 4: [25%] Consider a following single-server queuing system:

In the very beginning ($t = 0$), no packet is in the queue. Therefore, the queue length is $Q(0) = 0$.

At each time t, there are X_t number of packets arriving at the queue, where X_t is independently and identically distributed (i.i.d.) with $P(X_t = 2) = 1/3$ and $P(X_t = 0) = 2/3$. At time t, the server can serve "up to" Y_t packets, where Y_t is i.i.d. with $P(Y_t = 2) = 2/3$ and $P(Y_t = 0) = 1/3$. Since we cannot have a negative queue length, at the end of time t, the new queue length must be

$$Q(t) = \max(0, Q(t-1) + X_t - Y_t). \tag{5}$$

We are interested in the following quantity:

$$r \triangleq \lim_{T\to\infty} \frac{-1}{T} \log(\mathsf{Prob}(Q(T) \geq T)). \tag{6}$$

(That is, we are interested in finding the (exponential) decay rate of the overflow probability $\mathsf{Prob}(Q(T) \geq T) \approx e^{-rT}$.)

Question (20%): Use the Chernoff bound to find the value of r.

Hint 1: We first note that $Q(T) \neq \sum_{t=1}^{T}(X_t - Y_t)$.

Hint 2: You can directly use the following inequality without proving it

$$\mathsf{Prob}\left(\sum_{t=1}^{T}(X_t - Y_t) \geq T\right) \leq \mathsf{Prob}(Q(T) \geq T) \leq \sum_{k=1}^{T} \mathsf{Prob}\left(\sum_{t=k}^{T}(X_t - Y_t) \geq T\right). \tag{7}$$

Hint 3: You can further assume that the two terms $\mathsf{Prob}\left(\sum_{t=1}^{T}(X_t - Y_t) \geq T\right)$ and $\sum_{k=1}^{T} \mathsf{Prob}\left(\sum_{t=k}^{T}(X_t - Y_t) \geq T\right)$ have the same exponential decay rate without proving it.

Advanced Question (5%) Prove the above inequality (7).