Question 1: [25%] Consider the following transmission scheme:
Encoder: We use a binary linear code with the parity-check matrix being

$$H = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}. \tag{1}$$

Channel Model: We assume the following binary Markov channel:

Consider a sequence of Markov random variables $N_1, N_2, \cdots$. For the very first time instant ($t = 1$), we have $P(N_1 = 1) = 0.2$ and $P(N_1 = 0) = 0.8$, and

$$P(N_t = 1 | N_{t-1} = 0) = 0.1, \quad P(N_t = 0 | N_{t-1} = 0) = 0.9, \tag{2}$$
$$\text{and } P(N_t = 1 | N_{t-1} = 1) = 0.8, \quad P(N_t = 0 | N_{t-1} = 1) = 0.2 \tag{3}$$

for all $t = 2, 3, 4, \cdots$.

For any input bit X_t at time t, the output bit is $Y_t = X_t \oplus N_t$, the binary exclusive OR between X_t and N_t.

(This Markov channel model is widely used to model the "burst error".)

Question 1: If we observe $\vec{y}_{\text{obs}} = 100$, what is the most likely (maximum likelihood) codeword?

Question 2: What is the frame error rate of the maximum likelihood (ML) decoder?

$G = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow$ Two competing codewords 000 and 111.

$\vec{y}$ \ $\vec{x}$	000	111
000	0.8×0.9×0.9 ✓	0.2×0.8×0.8
001	0.8×0.9×0.1 ✓	0.2×0.8×0.2
010	0.8×0.1×0.2 ✓	0.2×0.2×0.1
011	0.8×0.1×0.8 ✓	0.2×0.2×0.9
100	0.2×0.2×0.9	0.8×0.1×0.8 ✓
101	0.2×0.2×0.1	0.8×0.1×0.2 ✓
1⌀0	0.2×0.8×0.2	0.8×0.9×0.1 ✓

111	0.2×0.8×0.8	0.8×0.9×0.9 ✓

$= \frac{1}{2} \times (0.2 + 0.2) = 0.2$ #

Question 2: [25%]

Consider the following binary-input binary-output channel: For any n-bit input string, let n_0 denote the number of 0s in this string and let n_1 denote the number of 1s in this string. (Obviously $n_0 + n_1 = n$.) If we pass this bit string through the given channel, then at most $n_1 \cdot p$ number of 1s will be corrupted and changed to 0. On the other hand, all 0 bits will remain intact. (You can view this channel as an approximation of the Z-channel mentioned in class.)

Suppose you are asked to design a code satisfying the following two conditions:

1. Each codeword has exactly $n \cdot \theta$ 1s and $n \cdot (1-\theta)$ 0s.
2. There is no decoding error, i.e., no two codewords can lead to the same corrupted observation $\vec{y}_{\text{obs}}$.

The rate of the code $R(n)$ is denoted by $R(n) \triangleq \frac{\log_2(\text{total number of codewords})}{n}$.

Question: Use the sphere-packing bound to upper bound R_∞, which is defined by

$$R_\infty \triangleq \lim_{n\to\infty} R(n). \tag{4}$$

Hint: You may need to use the Stirling's formula $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

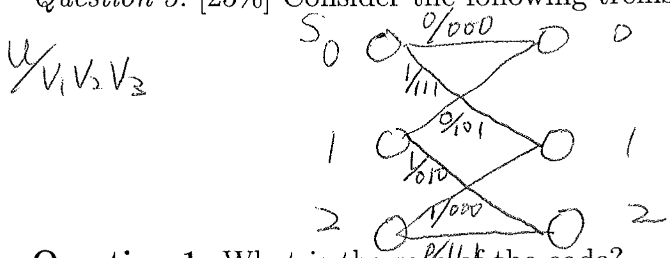
$$\frac{\binom{n}{n\theta\cdot(1-p)}}{\chi\cdot\binom{n\theta}{n\theta(1-p)}} \geq 1$$

$$\Rightarrow \quad \chi \leq \frac{\dfrac{n^n}{(n\theta(1-p))^{n\theta(1-p)}\,(n(1-\theta+\theta p))^{n(1-\theta+\theta p)}}}{\dfrac{(n\theta)^{n\theta}}{(n\theta(1-p))^{n\theta(1-p)}\,(n\theta p)^{n\theta p}}}$$

$$\Rightarrow \quad \frac{\log_2 \chi}{n} \leq \log_2\left(\frac{\cancel{\theta}(1-p)^{\theta(1-p)}\cdot p^{\theta p}}{(\theta(1-p))^{\theta(1-p)}\,(1-\theta+\theta p)^{(1-\theta+\theta p)}}\right)$$

$$\leq \boxed{\theta p\log_2(p) - \theta(1-p)\log_2(\theta(1-p)) - (1-\theta+\theta p)\log_2(1-\theta+\theta p)} \quad \#$$

Question 3: [25%] Consider the following trellis code structure:

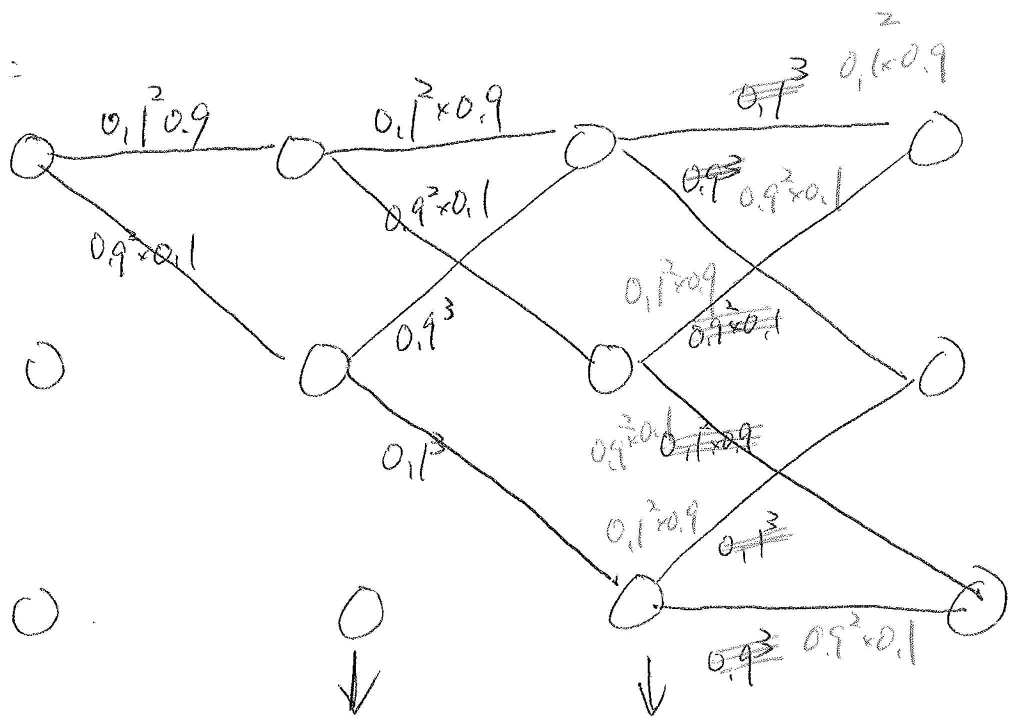


Question 1: What is the rate of the code?
Question 2: If the input bit string is $u_1u_2u_3$ =101, what is the output bit string?
Question 3: If we pass this code through a binary symmetric channel with crossover probability $p = 0.1$ and we observe y = 101101110. What is the most likely value of the second input bit u_2?

A1: $\frac{1}{3}$

A2: 1 1 1 1 0 1 1 1 1

A3:



Q	as	bs
	0,009	~~0.9³+0,1³ = 0.730~~ 0.09
	0,081	0,081+0,009 = 0,09
	0	0,9³+0,1³ = 0.730 . 0.09

$u_2 = 0$

$$0{,}009 \times \underline{0{,}1^2 \times 0{,}9} \times \overset{0{,}09}{\cancel{0{,}730}}$$

$$+\ 0{,}081 \times \underline{0{,}9^3} \times \overset{0{,}09}{\cancel{0{,}730}}$$

$$= \cancel{0{,}0431649}\ \ 0{,}053217$$

$u_2 = 1$

$$0{,}009 \times \underline{0{,}9^2 \times 0{,}1} \times 0{,}09$$

$$0{,}081 \times \underline{0{,}1^3} \times \cancel{0{,}730}\ \ 0{,}09$$

$$= \cancel{0{,}00012474}\ \ 7{,}29 \times 10^{-5}$$

$\Longrightarrow$ ML decision of $u_2 = 0$ #

Question 4: [35%] Consider a following single-server queuing system:

In the very beginning ($t = 0$), no packet is in the queue. Therefore, the queue length is $Q(0) = 0$.

At each time t, there are X_t number of packets arriving at the queue, where X_t is independently and identically distributed (i.i.d.) with $P(X_t = 2) = 1/3$ and $P(X_t = 0) = 2/3$. At time t, the server can serve "up to" Y_t packets, where Y_t is i.i.d. with $P(Y_t = 2) = 2/3$ and $P(Y_t = 0) = 1/3$. Since we cannot have a negative queue length, at the end of time t, the new queue length must be

$$Q(t) = \max(0, Q(t-1) + X_t - Y_t). \tag{5}$$

We are interested in the following quantity:

$$r \triangleq \lim_{T\to\infty} \frac{-1}{T} \mathsf{Prob}(Q(T) > T). \tag{6}$$

(That is, we are interested in finding the (exponential) decay rate of the overflow probability $\mathsf{Prob}(Q(T) > T) \approx e^{-rT}$.)

Question (25%): Use the Chernoff bound to find the value of r.

Hint 1: We first note that $Q(T) \neq \sum_{t=1}^{T}(X_t - Y_t)$.

Hint 2: You can directly use the following inequality without proving it

$$\begin{aligned} \mathsf{Prob}\left(\sum_{t=1}^{T}(X_t - Y_t) > T\right) &\leq \mathsf{Prob}(Q(T) > T) \\ &\leq \sum_{k=1}^{T} \mathsf{Prob}\left(\sum_{t=k}^{T}(X_t - Y_t) > T\right). \end{aligned} \tag{7}$$

Bonus questions (10%) Prove the above inequality.

$W_t = X_t - Y_t$

$$P(W_t = w) = \begin{cases} \frac{1}{9} & \text{if } w = 2 \\ \frac{4}{9} & \text{if } w = 0 \\ 4/9 & \text{if } w = -2 \end{cases}$$

$$P(W_t \geq 1) \leq \max_{s\geq 0} \frac{E(e^{sW_t})}{e^s}$$

$$= \max_{s \geq 0} \frac{\frac{1}{9}e^{2s} + \frac{4}{9} + \frac{4}{9} \times e^{-2s}}{e^s}$$

$$= \max_{s\geq 0} \frac{1}{9}e^{s} + \frac{4}{9}e^{-s} + \frac{4}{9} \times e^{-3s}.$$

Taking the derivative

$$\Rightarrow \frac{1}{9}e^{s} + (-1)\frac{4}{9}e^{-s} + (-3)\frac{4}{9}e^{-3s} = 0$$

$$\Leftrightarrow \frac{1}{9}e^{s}\left(1 - 4e^{-2s} - 12e^{-4s}\right) = 0.$$

$$\Leftrightarrow e^{-2s} = \frac{4 \pm \sqrt{16+48}}{-24}$$

$$= \underline{\frac{1}{6}} \text{ or } -\frac{1}{2}$$

$$\Rightarrow e^{s^*} = \sqrt{6}$$

$$\Rightarrow P(W_t \geq 1) \leq \frac{1}{9}\sqrt{6} + \frac{4}{9}\frac{1}{\sqrt{6}} + \frac{4}{9}\frac{1}{6\sqrt{6}}$$

$$= \frac{36\sqrt{6} + 24\sqrt{6} + 4\sqrt{6}}{9 \times 36}$$

$$= \frac{66\sqrt{6}}{9 \times 36} = \frac{11\sqrt{6}}{54}$$

$$\Rightarrow r = \log\left(\frac{54}{11\sqrt{6}}\right) \#$$

Bonus

Since

$$Q(t) - Q(t-1) = \max(-Q(t-1), X_t - Y_t) \geq X_t - Y_t$$

$$\Rightarrow \quad Q(T) \geq \sum_{t=1}^{T} (X_t - Y_t)$$

$$\Rightarrow P(Q(T) \geq T) \geq P\left(\sum_{t=1}^{T} (X_t - Y_t)\right)$$

Also. ~~if $Q(t) \geq T$ ... time~~ if $Q(T) \geq T$

let t^* be the ~~last ti~~ largest $t \leq T$

s.t. $Q(t^*) = 0$.

therefore $\forall t = t^*+1, \cdots, T$.

$$Q(t^*) > 0.$$

therefore

$$Q(T) - Q(t^*) = \sum_{t=t^*+1}^{T} X_t - Y_t \geq T$$

As a result, when $Q(T) \geq T$, there must exist a k s.t. $\sum_{t=k}^{T} (X_t - Y_t) \geq T$.

By the union bound, we have

$$P(Q(T) \geq T) \leq \cancel{\sum_{t=k}^{T} (X_t - Y_t) \geq T} \sum_{k=1}^{T} P\left(\sum_{t=k}^{T} (X_t - Y_t) \geq T\right) \quad \#$$