ECE 695C, Midterm #2

5:30–6:30pm Wednesday, April 12, 2012, EE 224,

1. Enter your name, student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. Some time-consuming and bonus questions are assigned a small number of points. Make sure you manage your time well during the exam.
4. Use the back of each page for rough work.
5. You can use a simple calculator. No help sheet is allowed.

Name:

Student ID:

E-mail:

Signature:

Question 1: [30%] Consider two Bernoulli variables X_1 and X_2. Each Bernoulli random variable is independently and uniformly distributed on $\{0, 1\}$. We also know that $Y_1 = (-1)^{X_1} + (-1)^{X_2} + N_1$ and $Y_2 = (-1)^{X_2} + N_2$ where N_1 and N_2 are independent standard Gaussian random variables (mean =0, variance =1). Suppose we observe $Y_1 = 0.9$ and $Y_2 = 0.5$. Find the maximal likelihood decision for the value of X_1.

Hint: You may like to use the factor graph representation to solve this problem.

Question 2: [30%]

For binary symmetric channels (BSCs), the Gallager's LDPC decoding algorithm B works as follows.

Each time, each variable/check node can send messages m to its neighbors. The message m takes one of the two values $\{-1, 1\}$.

The variable node message map is

$$m^{(0)} = \begin{cases} 1 & \text{if the received bit } Y_i = 0 \\ -1 & \text{if the received bit } Y_i = 1 \end{cases}$$

$$m_{\mathrm{v}\to\mathrm{c}} = \begin{cases} \text{if there is a majority of its incoming messages } m_1 \text{ to } m_{d_v-1}, \\ \quad \text{choose } m_{\mathrm{v}\to\mathrm{c}} \text{ to be the majority} \\ \text{if there is no majority (if there is a tie), choose } m_{\mathrm{v}\to\mathrm{c}} = m^{(0)} \end{cases} . \tag{1}$$

The check node message map is

$$m_{\mathrm{c}\to\mathrm{v}} = \prod_{i=1}^{d_c-1} m_i, \text{ the product of the incoming messages.} \tag{2}$$

Write down the density evolution formula of the Gallager's decoding algorithm B for the case of regular $(3, 4)$ LDPC codes (i.e., $d_v = 3$ and $d_c = 4$). You can assume that the crossover probability of the BSC is p.

Question 3: [30%] Consider an irregular (λ, ρ) code ensemble, for which the variable node degree distribution polynomial is $\lambda(x) = \sum_{k=2}^{\max d_v} \lambda_k x^{k-1}$ and the check node degree distribution polynomial is $\rho(x) = \sum_{k=2}^{\max d_c} \rho_k x^{k-1}$. By definition $\lambda'(0) = \frac{d}{dx}\lambda(x)\big|_{x=0} = \lambda_2$ and $\rho'(1) = \frac{d}{dx}\rho(x)\big|_{x=1}$.

1. [15%] Write down the EXIT curve formulas in terms of $\lambda(x)$ and $\rho(x)$.

2. [20%] Prove that for any binary erasure channel with erasure probability ϵ, if

$$1 < \epsilon \cdot \lambda'(0) \cdot \rho'(1), \tag{3}$$

   then the (λ, ρ) LDPC code cannot fully correct all erasures.

Hint: You should check whether there is an open tunnel by focusing on the top-right corner of the two curves (when both $I_{c \to v}$ and $I_{v \to c}$ are close to 1).

Question 4: [10 + 20% bonus] The classic belief propagation decoding for LDPC codes has the following message maps:

The variable node message map is

$$m^{(0)} = \log\left(\frac{P(Y|0)}{P(Y|1)}\right)$$

$$m_{\mathrm{v}\to\mathrm{c}} = m^{(0)} + \sum_{i=1}^{d_v-1} m_i.$$

The check node message map is

$$m_{\mathrm{c}\to\mathrm{v}} = 2\tanh^{-1}\left(\prod_{i=1}^{d_c-1} \tanh\left(\frac{m_i}{2}\right)\right).$$

From the lecture we know that the above message maps are equivalent to a BCJR-like decoding algorithm on the LDPC code.

Question:

1. [10%] Write down the Viterbi-like message passing decoding algorithm when focusing on a regular LDPC code with degrees d_v and d_c. Hint: Your answer should consist of the $\alpha_{v\to c}(x)$ message maps and $\beta_{v\to c}(x)$ message maps.

2. [Bonus 20%] Consider the following new set of message maps:

   The variable node message map is

$$m^{(0)} = \log\left(\frac{P(Y|0)}{P(Y|1)}\right)$$

$$m_{\mathrm{v}\to\mathrm{c}} = m^{(0)} + \sum_{i=1}^{d_v-1} m_i.$$

   The check node message map is

$$m_{\mathrm{c}\to\mathrm{v}} = \left(\prod_{i=1}^{d_c-1} \mathrm{sgn}(m_i)\right) \cdot \left(\min_{i=1,\cdots,d_c-1}(|m_i|)\right)$$

   where

$$\mathrm{sgn}(m_i) = \begin{cases} 1 & \text{if } m_i > 0 \\ 0 & \text{if } m_i = 0 \\ -1 & \text{if } m_i < 0 \end{cases}. \tag{4}$$

Prove that the above message maps correspond to the Viterbi-like decoding algorithm.

Hint: Your goal is to rewrite the message maps in the first sub-question by the log-likelihood-ratio messages $m_{v\to c} = \log\left(\frac{\alpha_{v\to c}(0)}{\alpha_{v\to c}(1)}\right)$ and $m_{c\to v} = \log\left(\frac{\beta_{c\to v}(0)}{\beta_{c\to v}(1)}\right)$.