35% Problem 1: Airy Stress Function, Principal Stresses, and Principal Directions

Given: Consider a thin rectangular panel loaded as shown in the figure below. The use of "thin rectangular panel" refers to a plane stress condition, where all stresses in the z-direction are zero. The following Airy stress function must be used to solve the problem:

Airy stress function: $\emptyset_{(x,y)} = c_1x^2 + c_2xy + c_3y^2$

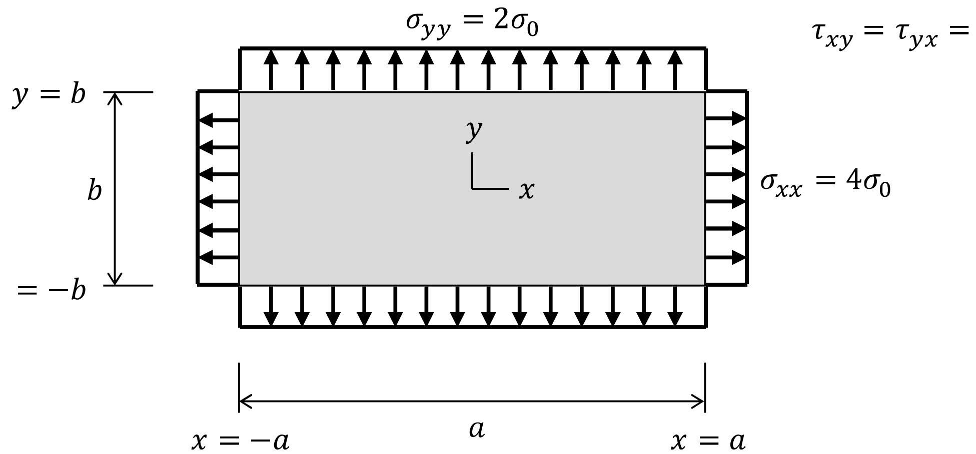


Figure: A thin rectangular panel

Required:

(1) Traction vectors on the left surface ($x = -a$) and the bottom surface ($y = -b$). (Please answer in terms of c_1, c_2, and c_3)

(2) Constants c_1, c_2, and c_3 in the Airy stress function $\emptyset_{(x,y)}$. (Please answer in terms of σ_0)

(3) Principal stresses σ_1 and σ_2 if $\sigma_0 = 1\ MPa$. Express your answers in "MPa."

(4) Principal directions θ_{p1} and θ_{p2} if $\sigma_0 = 1\ MPa$. Express your answers in "degree."

(For Parts 3 and 4, you may use either an eigen approach or Mohr's circle approach. Correct answers without proper justification will not receive a full credit.)

Solution: Write your solution below. Use the reverse side if necessary.

Solution

1.1

The Airy stress function is:

$$\Phi(x,y) = c_1x^2 + c_2xy + c_3y^2 \tag{1}$$

It is necessary to check the equilibrium equation and compatibility equation before using it.

- Check equilibrium: Automatically satisfied due to the use of an Airy stress function
- Check compatibility:
  - Order of the given $\Phi(x,y) = 2$
  - Order of compatibility equation $\nabla^2\nabla^2\Phi = 4$

Therefore, the compatibility equation is also satisfied.

Write σ_{xx}, σ_{yy}, τ_{xy} using the given $\Phi(x,y)$

$$\begin{aligned}
\sigma_{xx} &= \frac{\partial^2}{\partial y^2}\left(c_1x^2 + c_2xy + c_3y^2\right) = \frac{\partial}{\partial x}\left(c_2x + 2c_3y\right) = 2c_3 \\
\sigma_{yy} &= \frac{\partial^2}{\partial x^2}\left(c_1x^2 + c_2xy + c_3x^2\right) = \frac{\partial}{\partial x}\left(2c_1x + c_2y\right) = 2c_1 \\
\tau_{xy} &= -\frac{\partial^2}{\partial x \partial y}\left(c_1x^2 + c_2xy + c_2y^2\right) = -\frac{\partial}{\partial x}\left(c_2x + 2c_3y\right) = -c_2
\end{aligned} \tag{2}$$

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} 2c_3 & -c_2 \\ -c_2 & 2c_1 \end{bmatrix} \tag{3}$$

Applying B.C.: $\{t\} = [\sigma]\{n\}$

On the left surface ($x = -a$): $\{n\} = \begin{Bmatrix} -1 \\ 0 \end{Bmatrix}$

$$\{t\} = \begin{bmatrix} 2c_3 & -c_2 \\ -c_2 & 2c_1 \end{bmatrix}\begin{Bmatrix} -1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -2c_3 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} -4\sigma_0 \\ 0 \end{Bmatrix} \tag{4}$$

On the bottom surface ($y = -b$): $\{n\} = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix}$

$$\{t\} = \begin{bmatrix} 2c_3 & -c_2 \\ -c_2 & 2c_1 \end{bmatrix}\begin{Bmatrix} 0 \\ -1 \end{Bmatrix} = \begin{Bmatrix} c_2 \\ -2c_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -2\sigma_0 \end{Bmatrix} \tag{5}$$

1.2

From the B.C. given in the problem, we can get

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} 2c_3 & -c_2 \\ -c_2 & 2c_1 \end{bmatrix} \overset{\text{set}}{=} \begin{bmatrix} 4\sigma_0 & 0 \\ 0 & 2\sigma_0 \end{bmatrix} \tag{6}$$

Therefore, we can get

$$\left.\begin{aligned} 2c_3 = 4\sigma_0 &\rightarrow c_3 = 2\sigma_0 \\ -c_2 = 0 &\rightarrow c_2 = 0 \\ 2c_1 = 2\sigma_0 &\rightarrow c_1 = \sigma_0 \end{aligned}\right\} \Phi(x,y) = \sigma_0x^2 + 2\sigma_0y^2 \tag{7}$$

Check:

$$
\begin{aligned}
\sigma_{xx} &= \frac{\partial^2\Phi}{\partial y^2} = \frac{\partial^2}{\partial y^2}\left(\sigma_0 x^2 + 2\sigma_0 y^2\right) = 4\sigma_0 \\
\sigma_{yy} &= \frac{\partial^2\Phi}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(\sigma_0 x^2 + 2\sigma_0 y^2\right) = 2\sigma_0 \\
\tau_{xy} &= -\frac{\partial^2\Phi}{\partial x\partial y} = -\frac{\partial^2}{\partial x\partial y}\left(\sigma_0 x^2 + 2\sigma_0 y^2\right) = -\frac{\partial}{\partial x}\left(4\sigma_0 y\right) = 0
\end{aligned} \tag{8}
$$

1.3

Using Mohr's circle:

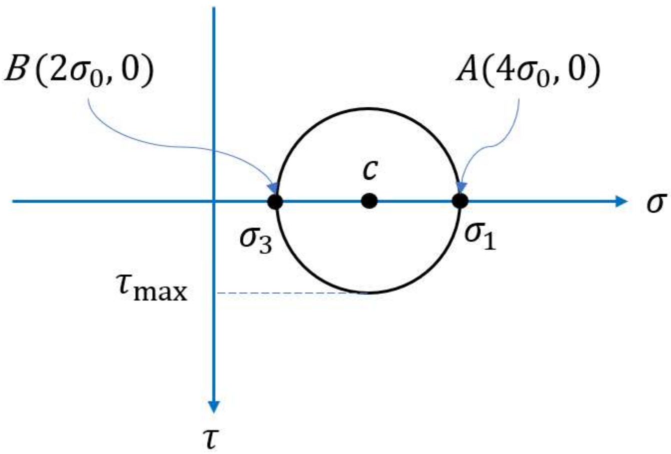


Figure 1: The Column Mohr's circle

$$
\begin{aligned}
c &= (\sigma_{avg}, 0) = (3\sigma_0, 0) \\
R &= \tau_{\max} = \sqrt{(\sigma_0)^2 + (0)^2} = \sigma_0 \\
\sigma_1 &= \sigma_{avg} + R = 3\sigma_0 + \sigma_0 = 4\sigma_0 = A \\
&\rightarrow \sigma_1 = A \\
\sigma_2 &= \sigma_m - R = 3\sigma_0 - \sigma_0 = 2\sigma_0 = B \\
&\rightarrow \sigma_2 = B
\end{aligned} \tag{9}
$$

Based on the analysis above,

$$
\begin{aligned}
&\sigma_1 = 4\sigma_0, \text{ where } \sigma_0 = 1\,\text{MPa} \\
&\sigma_1 = 4\,\text{MPa} \\
&\sigma_2 = 2\sigma_0 \\
&\sigma_2 = 2\,\text{MPa}
\end{aligned} \tag{10}
$$

1.4

Since $\sigma_1 = A, \quad \sigma_2 = B$, we have

$$
\begin{aligned}
&\theta_{p_1} = 0 \\
&\theta_{p_2} = \theta_{p_1} + 90^\circ = 90^\circ
\end{aligned} \tag{11}
$$

Problem 1	credit
9 answers are correct 1) Two traction vectors 2) Three constants 3) Two principal stresses 4) Two principal direction angles	35
8 answers are correct	33
7 answers are correct	31
6 answers are correct	29
5 answers are correct	27
4 answers are correct	25
3 answers are correct	23
2 answers are correct	21
1 answer are correct	19
0 answer are correct. The solution procedure has minor issues	17
0 answer are correct. The solution procedure has major issues.	12
0 answer are correct. Good faith effort	8
blank	0

Note: If students did not check equilibrium and compatibility equations before using Airy stress function. A warning would be given. We did not take any points off for this exam.

<u>35% Problem 2: 3D Stress-Strain Relations</u>

Given: A specimen is subjected to a compressive stress σ_{zz} and is confined in a rigid fixture so that the specimen cannot deform in the y direction (Fig. 2). However, deformation is permitted in the x-direction. The containing walls are smooth (i.e., frictionless). Assume that the specimen is isotropic, exhibits linear-elastic behavior, and has Young's modulus (E) and Poisson's ratio (ν). Since the specimen is not permitted to deform in the y-direction, $\varepsilon_{yy} = 0$. Also, $\sigma_{xx} = 0$ since the specimen is permitted to deform in the x-direction freely.

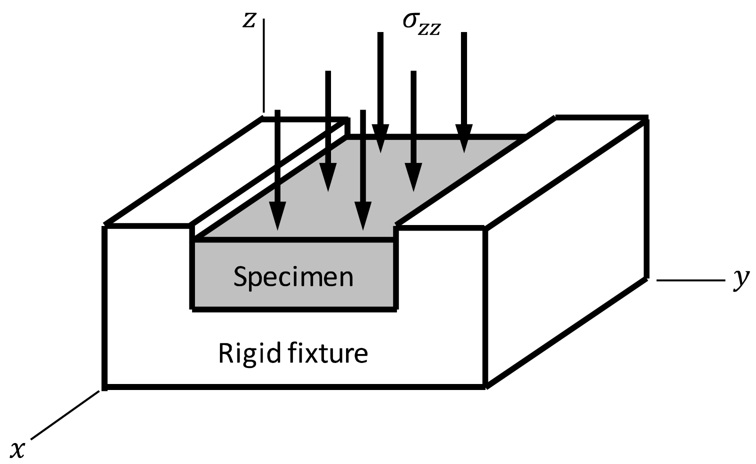


Figure 2: A specimen in a rigid fixture

<u>Required:</u>

Hint: Start solving the problem with the following expression: $\{\varepsilon\} = [a]\{\sigma\}$

(1) Calculate σ_{yy} in terms of σ_{zz}, E, and/or ν

(2) Calculate ε_{zz} in terms of σ_{zz}, E, and/or ν

(3) Calculate ε_{xx} in terms of σ_{zz}, E, and/or ν

(4) Calculate $\Delta V/V$ if σ_{zz} = 100 MPa, E = 70 GPa, and ν = 0.33
(Note: $\Delta V/V$ is the ratio of the change in volume to the original volume)

<u>Solution:</u> Write your solution below. Use the reverse side if necessary.

Solution:

List the constitutive relation: $\{\varepsilon\} = [a]\{\sigma\}$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1/E & -v/E & -v/E & 0 & 0 & 0 \\ -v/E & 1/E & -v/E & 0 & 0 & 0 \\ -v/E & -v/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}$$

And by the boundary condition, $\varepsilon_{yy} = \sigma_{xx} = 0$

The constitutive equations can be written as:

$$-\frac{v}{E}\sigma_{yy} - \frac{v}{E}\sigma_{zz} = \varepsilon_{xx}$$

$$\frac{1}{E}\sigma_{yy} - \frac{v}{E}\sigma_{zz} = 0$$

$$-\frac{v}{E}\sigma_{yy} + \frac{1}{E}\sigma_{zz} = \varepsilon_{zz}$$

Solve linear equations

$\sigma_{yy} = v\sigma_{zz}$, $\varepsilon_{xx} = -\frac{v(1+v)}{E}\sigma_{zz}$, and $\varepsilon_{zz} = \frac{(1-v^2)}{E}\sigma_{zz}$

For small deformation, $\frac{\Delta V}{V} \cong \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = -\frac{v(1+v)}{E}\sigma_{zz} + 0 + \frac{(1-v^2)}{E}\sigma_{zz}$=-6.46e-4 or 6.46e-4 as some students may directly plug in $\sigma_{zz} = 100MPa$ instead of $\sigma_{zz} = -100MPa$.

Grading Rubric

Problem 2 (35%)

Constitutive relation or linear equations (14%)

Realize $\varepsilon_{yy} = \sigma_{xx} = 0$ (3%)

Process of solving constitutive relation or linear equations (6%) which leads to $\sigma_{yy} = -v\sigma_{zz}$, $\varepsilon_{xx} = \frac{v(1+v)}{E}\sigma_{zz}$, and $\varepsilon_{zz} = -\frac{(1-v^2)}{E}\sigma_{zz}$ (1%) each.

$\frac{\Delta V}{V} \cong \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$ (6%)

Plug in all correct numerical values into $\frac{\Delta V}{V} \cong \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{v(1+v)}{E}\sigma_{zz} + 0 - \frac{(1-v^2)}{E}\sigma_{zz}$ (2%)

Final value of $\frac{\Delta V}{V}$ is correct (1%).

Prob. 2 Extra:

Some students use lame parameters and make the computation process complicated. The last page shows the process of solving the problem with stiffness matrix by lame parameters.

The grading policy is that if you can still find the following relation:

$$\varepsilon_{zz} = \frac{(\lambda + 2G)}{4G(G + \lambda)}\sigma_{zz}$$

$$\varepsilon_{xx} = \frac{-\lambda}{4G(G + \lambda)}\sigma_{zz}$$

$$\sigma_{yy} = \frac{\lambda}{2(G + \lambda)}\sigma_{zz}$$

And states the relation between λ, G and E, v.

$\lambda = \frac{Ev}{(1+v)(1-2v)}$ and $G = \frac{E}{2(1+v)}$

Or whatever your answers are and the answers could lead to

$$\varepsilon_{zz} = (1.273e - 11)\sigma_{zz}$$

$$\varepsilon_{xx} = (-6.27e - 11)\sigma_{zz}$$

$$\sigma_{yy} = 0.33\sigma_{zz}$$

if the value of E=70GPa, and v=0.33 are plugged in. You can still get the full score. Grading rubric are the same as the original. Just the constitutive equation and linear equation may look different. But as the problem itself has given hint that you should use $\{\varepsilon\} = [a]\{\sigma\}$, lame parameters is not encouraged here.

$$\begin{Bmatrix} \cancel{\sigma_{xx}}^{\,0} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \lambda+2G & \lambda & \lambda & & & \\ \lambda & \lambda+2G & \lambda & & & \\ \lambda & \lambda & \lambda+2G & & & \\ & & & G & & \\ & & & & G & \\ & & & & & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \cancel{\varepsilon_{yy}}^{\,0} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

$\varepsilon_{zz} = 1.273 \times 10^{-11} \sigma_{zz}$

$\varepsilon_{xx} = -6.27 \times 10^{-12} \sigma_{zz}$

$$(\lambda+2G)\varepsilon_{xx} + \lambda \varepsilon_{zz} = 0. \quad \text{—①}$$

$$\lambda \varepsilon_{xx} + \lambda \varepsilon_{zz} = \sigma_{yy} \quad \text{—②}$$

$$\lambda \varepsilon_{xx} + (\lambda+2G)\varepsilon_{zz} = \sigma_{zz}. \quad \text{—③}$$

①. $\lambda \varepsilon_{zz} = -(\lambda+2G)\varepsilon_{xx} \longrightarrow \varepsilon_{zz} = -\dfrac{(\lambda+2G)}{\lambda}\varepsilon_{xx}$

$$\boxed{\varepsilon_{zz} = \frac{(\lambda+2G)}{4G(G+\lambda)}\sigma_{zz}}$$

$$\lambda \varepsilon_{xx} + (\lambda+2G)\varepsilon_{zz} = \sigma_{zz}.$$

$$\rightarrow \lambda \varepsilon_{xx} - \frac{(\lambda+2G)^2}{\lambda}\varepsilon_{xx} = \sigma_{zz}.$$

$$\rightarrow \frac{\lambda^2 - (\lambda+2G)^2}{\lambda}\varepsilon_{xx} = \sigma_{zz}.$$

$$\rightarrow \varepsilon_{xx} = \underbrace{\frac{\lambda}{\lambda^2 - (\lambda^2+2G)^2}}_{4\lambda G + 4G^2} \sigma_{zz} = \boxed{-\frac{\lambda}{4G(G+\lambda)}\sigma_{zz}}$$

②. $\lambda \varepsilon_{xx} + \lambda \varepsilon_{zz} = \sigma_{yy}$

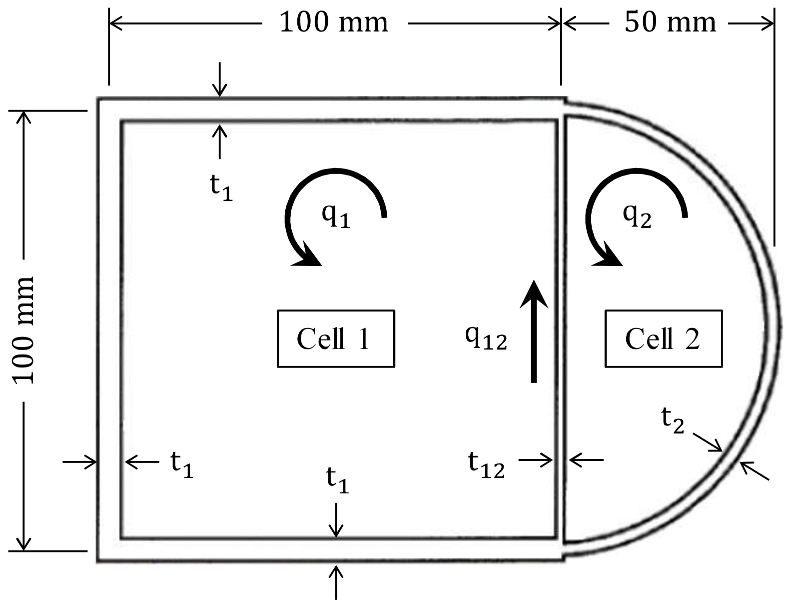
$$\rightarrow \lambda \varepsilon_{xx} + \cancel{\lambda}\left(-\left(\frac{\lambda+2G}{\cancel{\lambda}}\right)\right)\varepsilon_{xx} = \sigma_{yy}.$$

$$\rightarrow \lambda \varepsilon_{xx} - \lambda\varepsilon_{xx} - 2G\varepsilon_{xx} = \sigma_{yy} \rightarrow -2G\varepsilon_{xx} = \sigma_{yy}.$$

$$\rightarrow \underline{\sigma_{yy} = -2G \cdot \frac{\lambda}{-4\lambda G - 4G^2}\sigma_{zz} = \boxed{\frac{\lambda}{2\lambda+2G}\sigma_{zz}}}$$

30% Problem 3: Torsion in a Thin-Walled Multi-Cell Cross Section

Given:



$t_1 = 3$ mm
$t_2 = 2$ mm
$t_{12} = 1$ mm

Figure 1: Cross section of a thin-walled multi-cell torsion member

Material properties: E = 72 GPa, ν = 0.33, and $G = \frac{E}{2(1+\nu)}$

Applied torque: $T = 5 \times 10^5$ N · m

Required:

(1) Areas enclosed by the center lines ($\overline{A}_1$ and $\overline{A}_2$).
(2) Shear flows in the walls (q_1, q_2, and q_{12}).
(3) Maximum shear stress (τ_{max}).
(4) Angle of twist per unit length (θ).

Solution:

Set

q_1 = Shear flow on the left cell

q_2 = Shear flow on the right cell

$q_{12} = q_1 - q_2$ = Shear flow in the vertical web

$T = 2\bar{A}_1 q_1 + 2\bar{A}_2 q_2$, where $\bar{A}_1 = 100\ mm\ \times 100\ mm = .01\ m^2$ & $\bar{A}_2 = \frac{1}{2}\frac{\pi d^2}{4} = 0.0039\ m^2$

$G = \frac{E}{2(1+\nu)} = 27.07$ GPa

Left Cell:

$$\theta_1 = \frac{1}{2G\bar{A}_1}\oint \frac{q}{t} ds = \frac{1}{2G\bar{A}_1}\left\{\frac{s_1 q_1}{t_1} + \frac{s_{12} q_{12}}{t_{12}}\right\} = 1.8472 * 10^{-7}(q_1 + q_{12})$$

Right Cell:

$$\theta_2 = \frac{1}{2G\bar{A}_2}\oint \frac{q}{t} ds = \frac{1}{2G\bar{A}_2}\left\{\frac{s_2 q_2}{t_2} - \frac{s_{12} q_{12}}{t_{12}}\right\} = 3.72 * 10^{-7} q_2 - 4.7365 * 10^{-7} q_{12}$$

Since the entire thin-wall section must rotate as a rigid body in the plane, we require the compatibility condition: $\theta_1 = \theta_2$

$$q_{12} = q_1 - q_2$$

$$q_1 = 1.22214 q_2$$

Plug back into θ_1, and let $\theta_1 = \theta_2 = \theta$

$$\theta = 2.66786 \times 10^{-7} q_2$$

$$q_2 = 3.74832 \times 10^6\ \theta$$

$$q_1 = 4.58098 \times 10^6\ \theta$$

$$q_{12} = 8.3266 \times 10^5\ \theta$$

Since applied torque $T = 5 \times 10^5\ \mathrm{N \cdot m}$,

$$T = 2\bar{A}_1 q_1 + 2\bar{A}_2 q_2 = 2(0.01)(1.22214 q_2) + 2(0.0039) q_2 = 5 \times 10^5\ \mathrm{N \cdot m}$$

$$q_2 = 1.54785 \times 10^7$$

$$q_1 = 1.89522 \times 10^7$$

$$q_{12} = 3.4384 \times 10^6$$

Then,

$$\theta = 2.66786 \times 10^{-7} q_2 = 4.12945\ \frac{rad}{m} \quad or\ 236.6 \frac{deg}{m}$$

Or, we can solve for θ first,

$$T = 2\bar{A}_1 q_1 + 2\bar{A}_2 q_2 = 2(0.01)(4.58098 \times 10^6\ \theta) + 2(0.0039)(3.74832 \times 10^6\ \theta)$$
$$= 5 \times 10^5\ \mathrm{N \cdot m}$$

$$\theta = 4.13714 \frac{rad}{m}\ \ or\ 237.041 \frac{deg}{m}$$

Then,

$$q_2 = 1.55073 \times 10^7\ N - m$$

$$q_1 = 1.89522 \times 10^7\ N - m$$

$$q_{12} = 3.44483 \times 10^6\ N - m$$

Either one is fine.

Now, we can determine $\tau_{max} = \frac{q}{t}$

$$\tau_{max1} = \frac{q_1}{t_1} = \frac{1.89522 \times 10^7}{.003} = 6.3174\ GPa$$

$$\tau_{max2} = \frac{q_2}{t_2} = \frac{1.55073 \times 10^7}{.002} = 7.75365\ GPa$$

$$\tau_{max3} = \frac{q_{12}}{t_{12}} = \frac{3.44483 \times 10^6}{.001} = 3.44483\ GPa$$

Therefore, looking for the maximum value of τ_{max}, $\tau_{max} = \tau_{max2} = 7.75365\ GPa$

Grading Rubric

Problem 3 (30%)

Problem setup (E, G, ν, thickness, and T) (5)

T equation (2)

$\overline{A}_1$ and $\overline{A}_2$ (2)

θ_1 and θ_2 integrals (4)

Compatibility eqn. (4)

q_1 and q_2 and q_{12} values (3)

θ values (2)

$\tau_{max1}, \tau_{max2}, \tau_{max3}$ eqns (3) and values (3)

Determine τ_{max} (2)

If students made mistakes in the first place (like mistakes due to calculation errors) but did procedure correctly (due to wrong G or area, wrong θ_1, θ_2, q_1, q_2, q_{12}, τ_{max}), then take 10% of the total grade (3%) off. Don't take off every half point whenever you see the wrong final answer for each answer. And if student made mistakes in the second place (due to wrong relations between q_1 and q_2, wrong final answers), then take 5% of total grade off (1.5%).